Thursday, April 3, 2008

Differential Equations and Dynamical Systems.
by Lawrence Perko. Texts in Applied Mathematics: 7

CHAPTER 1 Linear Systems

Lunes, Abril 8, 2008

This chapter presents the resolution of the linear system:

$$\dot{x} = Ax,$$

where: $x \in \mathbb{R}^n$, A is an $n \times n$ matrix and $\dot{x} = \frac{dx}{dt}$.

The solutions, as it will be shown, are given by:

$$x(t) = e^{At}\, x(0),$$

where the goal will be to define the exponential matrix e^{At}.

We will also study the behavior of the solutions of this system in 2-dimensions, and describe what could happen in more dimensions.

The study of the solutions will be given in terms of the eigenvalues of the matrix A.

$\boxed{1.1}$ If we start with the uncoupled system:

Section 1.1
Uncoupled Systems

$$\dot{x}_1 = -x_1,$$
$$\dot{x}_2 = 2x_2,$$

it is easy to find the solution:

$$x_1(t) = e^{-t} x_1(0)$$
$$x_2(t) = e^{2t} x_2(0)$$

This system can be written as:

$$\dot{x} = A x \quad , \text{with} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

with solution:

$$\vec{x}(t) = \begin{pmatrix} e^{-t} & \\ & e^{2t} \end{pmatrix} \vec{x}(0),$$

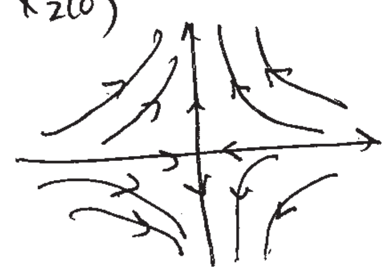
Here:

$$e^{At} \equiv \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}$$

Note that the solution satisfies:

$$x_1^2(2)\, x_2(t) = x_1^2(0)\, x_2(0)$$

which are hyperbolas in the x_1-x_2-plane.



(Tues, April 8, 2008.

* The x_1x_2-plane is the phase-plane.
The arrows indicate the direction of motion.

* For $x_1(0) = x_2(0) = 0$, the solution is:

$$x_1(t) = 0$$
$$x_2(t) = 0,$$

which is referred to as an equilibrium point.
(stationary solution,
equilibrium solution,
fixed point)

* The phase portrait of a system of diff. eq.: $\dot{x} = Ax$, is the set of all solution curves in the phase space $\mathbb{R}^n$.

* The dynamical system defined by the equation $\dot{x} = Ax$ is the mapping:

$$\phi : \mathbb{R} \times \mathbb{R}^2 \longmapsto \mathbb{R}^2$$

defined by its solution:

$$\phi(t, \vec{c}) = \vec{x}_c(t),$$

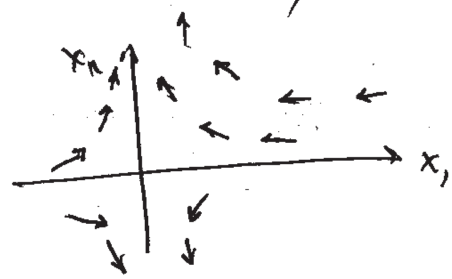
where $\vec{c} = (x_1(0), x_2(0))^t$ is the initial condition.

* **Geometrically**, the dynamical system describes the motion of the points in the phase plane along the solution curves defined by the system of the differential equations.

The function:

$$\vec{f}(\vec{x}) = A\bar{x}$$

defines a vector map. We can compute the vector field by small arrows at $\bar{x}$, with derivatives $\dot{x} = \vec{f}(x)$:



Example 3-D system.

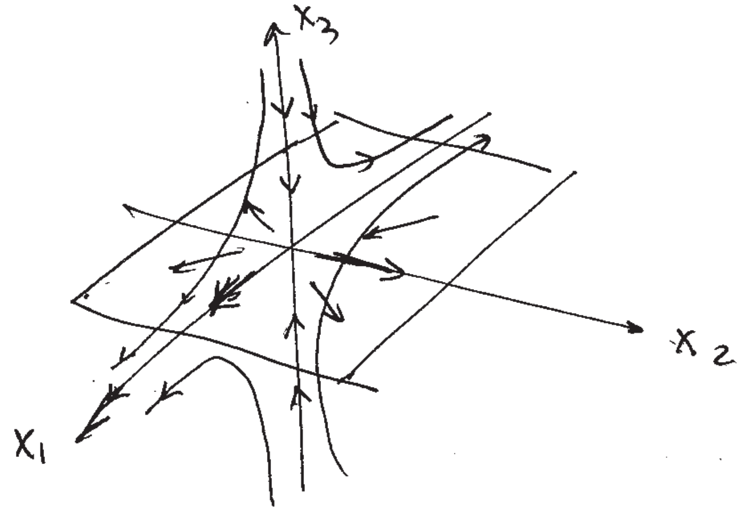
$$\dot{x}_1 = x_1$$

(3) .... $\dot{x}_2 = x_2$ has solution.

$$\dot{x}_3 = -x_3$$

$$x_1(t) = e^{t}\, x_1(0)$$
$$x_2(t) = e^{t}\, x_2(0)$$
$$x_3(t) = e^{-t}\, x_2(0).$$

The solution behaves as in the figure, next page



Notice that in the $x_1 x_2$-plane, the solutions goes away. Then, it is the unstable subspace.

On the other hand, along the x_3-axis, the solutions go inside the origin. Then, x_3-axis is the stable subspace of the system (3).

Section 1.2. **Diagonalization.**

Here, we use the following Theorem from Linear Algebra:

Theorem: If the eigenvalues of an $n \times n$ matrix A are real and distinct: $\lambda_1, \lambda_2, \ldots, \lambda_n$, with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then, this set of eigenvectors from a basis of $\mathbb{R}^n$, the matrix

$$P = (v_1\ v_2\ \cdots\ v_n)$$

is invertible, and

$$A = P \Lambda P^{-1},$$

with
$$\Lambda \equiv \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

This theorem can help us to find solutions of linear systems by a reduction to uncoupled system.

Consider a matrix A satisfying the conditions of the previous theorem, and let it define the system of ode's:
$$\dot{x} = Ax,$$

Then, define
$$\dot{x} = P \Lambda P^{-1} x.$$

and
$$(P^{-1}x)^{\cdot} = \Lambda (P^{-1}x)$$

i.e.
$$\dot{y} = \Lambda y,$$

where $y \equiv P^{-1}x$. This system has a solution:
$$y(t) = e^{\Lambda t} y(0),$$

where $e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix}$, and $y(0) = P^{-1}x(0)$.

We could solve the equation for y, since it was uncoupled. Lunes, April 7, 2005

Now; $y(t) = e^{\Lambda t} y(0)$, implies

$$P^{-1} x(t) = e^{\Lambda t} P^{-1} x(0)$$

$$\Rightarrow \quad \boxed{\vec{x}(t) = P e^{\Lambda t} P^{-1} \vec{x}(0)}$$

is the solution of $\dot{x} = Ax$, with initial cond. $x(0)$

Example:

$$\dot{x}_1 = -x_1 - 3x_2$$

$$\dot{x}_2 = 2x_2 .$$

The matrix here is:

$$A = \begin{pmatrix} -1 & -3 \\ 0 & -2 \end{pmatrix},$$

with eigenvalues:

$$\det(A - \lambda I) = 0; \quad \begin{pmatrix} -(1+\lambda) & -3 \\ 0 & -(2+\lambda) \end{pmatrix} = 0 .$$

$$(1+\lambda)(2+\lambda) = 0 \quad \Rightarrow \quad \boxed{\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -2 \end{array}}$$

and eigenvectors $\quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

The matrix P is: $\quad P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \; P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

The solution to the system is given by:

$$\bar{X}(t) = P\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1} X(0).$$

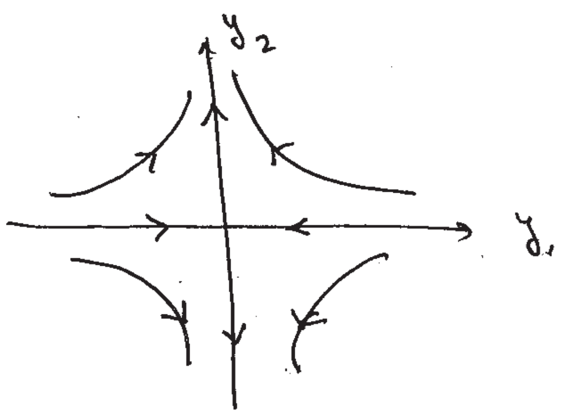
or

$$x(t) = \begin{pmatrix} & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} & 1 \\ 0 & 1 \end{pmatrix} x(0)$$

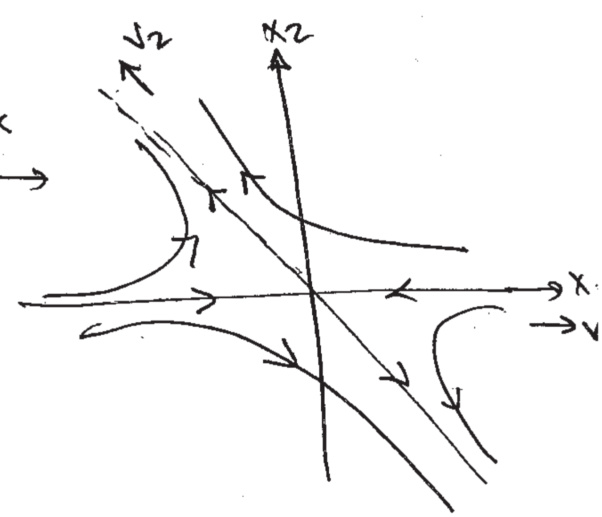
$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & e^{-t} \\ 0 & e^{2t} \end{pmatrix} x(0)$$

$$= \begin{pmatrix} e^{-t} & e^{-t} - e^{2t} \\ 0 & e^{-t} \end{pmatrix} x(0)$$

Then: $$\vec{X}(t) = \begin{pmatrix} x_1(0)e^{-t} + x_2(0)(e^{-t} - e^{2t}) \\ x_2(0)e^{-t} \end{pmatrix}$$



$\xrightarrow{Py = x}$



The phase portrait is as above;

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If $\lambda_1, \lambda_2, \ldots, \lambda_n$ all are real, and distinct.

$$\lambda_1, \lambda_2, \ldots, \lambda_k > 0$$

$$\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n < 0,$$

Let $(v_1, v_2, \ldots, v_n)$ be the corresponding eivectors.

$E^s = \mathrm{Span}\{v_1, \ldots, v_k\}$ - stable subspace

$E^u = \mathrm{Span}\{v_{k+1}, \ldots, v_n\}$ - unstable subspace.

For complex eivals, we also have a center subspace.
(see section 1.9)

$\boxed{1.3}$ The Exponential of an operator:

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$,

be a linear operator, $T \in \mathcal{L}(\mathbb{R}^n)$.

Operator norm: $\|T\| = \max_{|x| \leq 1} |T(x)|$.

where: $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. It has the properties; $T \in \mathcal{L}(\mathbb{R}^n)$, $S \in \mathcal{L}(\mathbb{R}^n)$.

1) $\|T\| \geq 0$

$\|T\| = 0 \Leftrightarrow T \equiv 0$

2) $\|kT\| = |k| \|T\|$, $k \in \mathbb{R}$

 =259= * 3) $\|S + T\| \leq \|S\| + \|T\|$.

If A represents the transformation T as a matrix,

the
$$\|A\| \le \sqrt{n}\, l,$$
where l is the maximum length of rows of A.

Proof: Let $A = \begin{pmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \vdots \\ \tilde{A}_n^T \end{pmatrix}$, where $\tilde{A}_j^T$ is the j column of A.

Then $Ax = \begin{pmatrix} \tilde{A}_1^T x \\ \vdots \\ \tilde{A}_n^T x \end{pmatrix}$

and
$$\begin{aligned}
\|Ax\|^2 &= |\tilde{A}_1^T x|^2 + \cdots + |\tilde{A}_n^T x|^2 \\
&\le \|\tilde{A}_1^T\|^2 \|x\|^2 + \cdots + \|\tilde{A}_n^T\|^2 \|x\|^2, \text{ by Cauchy-Schwartz} \\
&= \left(\|A_1^T\|^2 + \|A_2^T\|^2 + \cdots + \|\tilde{A}_n^T\|^2\right) \|x\|^2. \\
&\le n l \|x\|^2 \le n l^2, \text{ since } \|x\|^2 \le 1
\end{aligned}$$

where $l^2 = \max\{\|A_1^T\|^2, \ldots, \|\tilde{A}_n^T\|^2\}$.

Then:
$$\|Ax\| \le \sqrt{n}\, l.$$

$$\Rightarrow \max_{\|x\| \le 1} \|Ax\| \le \sqrt{n}\, l \Rightarrow \boxed{\|A\| \le \sqrt{n}\, l}$$

QED.

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<u>Definition</u> Consider the seq. $T_k \in \mathcal{L}(\mathbb{R}^n)$.

The seq. T_k converges to $T \in \mathcal{L}(\mathbb{R}^n)$ if, given $\epsilon > 0$, there is an $N(\epsilon)$, such that.

if $k \geq N(\epsilon)$, then $\|T - T_k\| < \epsilon$;

and we write:

$$\lim_{k \to \infty} T_k = T$$

<u>Lemma</u>: $S, T \in \mathcal{L}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

(1) $\|T(x)\| \leq \|T\| \, \|x\|$.

(2) $\|TS\| \leq \|T\| \, \|S\|$.

(3) $\|T^k\| \leq \|T\|^k$, for any $k \in \mathbb{N}$.

<u>Proof</u>: See text.

<u>Theorem</u>: Given $T \in \mathcal{L}(\mathbb{R}^n)$, and $t_0 > 0$, the series:

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}.$$

is absolutely and uniformly convergent for $|t| \leq t_0$.

<u>Proof</u>: See text.

We then can define:

$$e^T \equiv \sum_{k=0}^{\infty} \frac{T^k}{k!}.$$

By properties of limits:

(i) e^T is a linear operator.

(ii) $\|e^T\| \leq e^{\|T\|}$, from proof in text.

This allows to define:

<u>Definitions</u> Let A be $n \times n$ matrix. For $t \in \mathbb{R}$:

$$e^{At} \equiv \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

This definition will be useful in the definition of the solutions to $\dot{x} = Ax$.

We will compute e^{At} in terms of the eigenvalues and eigenvectors of A.

<u>Proposition 1</u>: If $S, T \in \mathcal{L}(\mathbb{R}^n)$,

$$S = PTP^{-1},$$

then $$e^S = P e^T P^{-1}.$$

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Proof:

$$e^S = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{S^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(PTP^{-1})^k}{k!} =$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{PT^kP^{-1}}{k!} = \lim_{n\to\infty} \left[P\left(\sum_{n=0}^{n} \frac{T^k}{k!}\right)P^{-1}\right]$$

$$= P\left(\lim_{n\to\infty} \sum_{n=0}^{n} \frac{T^k}{k!}\right)P^{-1} = P e^T P^{-1}.$$

QED.

Corollary. If A is $n\times n$, with n distinct eivals: $\lambda_1, \lambda_2, \ldots, \lambda_n$, with eivectors $(v_1\ v_2 \cdots v_n) = P$, then:

$At = P\Lambda t P^{-1}$ implies $\underline{e^{At} = P e^{\Lambda t} P^{-1}}$.

I.e. If $A = P\Lambda P^{-1} \Rightarrow e^{At} = P e^{\Lambda t} P^{-1}$.

Proposition 2 If $S, T \in \mathcal{L}(\mathbb{R}^n)$, s.t. $[S,T] = 0$, i.e. $ST - TS = 0$ they commute, then $e^{S+T} = e^S e^T$.

Proof: Check text. Verify that:

$$\sum_{n=0}^{\infty} \sum_{j+k=n} \frac{S^jT^k}{j!k!} = \sum_{j=0}^{\infty} \frac{S^j}{j!} \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

Corollary 2. If T is a linear transformation on $\mathbb{R}^n$.

$$(e^T)^{-1} = e^{-T}.$$

Corollary 3 If $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

then:

$$e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}.$$

Proof Let $\lambda = a + ib$.
$\gamma = \alpha + i\beta$.

Then:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} a\alpha - b\beta & -a\beta - b\alpha \\ b\alpha + a\beta & -b\beta + a\alpha \end{pmatrix}$$

Notice that $\lambda\gamma = (a+ib)(\alpha + i\beta)$
$= (a\alpha - b\beta) + i(a\beta + b\alpha)$

$$= \begin{pmatrix} \mathrm{Re}(\lambda\gamma) & -\mathrm{Im}(\lambda\gamma) \\ \mathrm{Im}(\lambda\gamma) & \mathrm{Re}(\lambda\gamma) \end{pmatrix}.$$

If $a = \alpha$, $\beta = b \Rightarrow \lambda = \gamma$ and:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} \mathrm{Re}(\lambda^2) & -\mathrm{Im}(\lambda^2) \\ \mathrm{Im}(\lambda^2) & \mathrm{Re}(\lambda^2) \end{pmatrix}.$$

Assume:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k = \begin{pmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{pmatrix}$$

Let $\gamma \equiv Re(\lambda^k) + i\, Im(\lambda^k)$ $\quad \begin{cases} \alpha = Re(\lambda^k) \\ \beta = Im(\lambda^k) \end{cases}$

Then

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{k+1} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} Re(\lambda\gamma) & -Im(\lambda\gamma) \\ Im(\lambda\gamma) & Re(\lambda\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} Re(\lambda^{k+1}) & -Im(\lambda^{k+1}) \\ Im(\lambda^{k+1}) & Re(\lambda^{k+1}) \end{pmatrix}.$$

Then, as in the text:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\begin{pmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} Re\left(\frac{\lambda^k}{k!}\right) & -Im\left(\frac{\lambda^k}{k!}\right) \\ Im\left(\frac{\lambda^k}{k!}\right) & Re\left(\frac{\lambda^k}{k!}\right) \end{pmatrix}$$

$$= \begin{pmatrix} Re(e^\lambda) & -Im(e^\lambda) \\ Im(e^\lambda) & Re(e^\lambda) \end{pmatrix} = \begin{pmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{pmatrix}$$

=265= =15=.

To finally get:

$$\boxed{e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}.}$$

Corollary 4. For

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

then

$$e^A = e^a \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Proof: Notice: $A = aI + B$, with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$

Notice that $[I, B] = 0$.

Then: $e^A = e^{aI+B} = e^{aI}e^B = e^a e^B$.

Then: Now: $e^B = I + B$ (since $B^2 = \dots = B^n = \dots = 0$)

Then

$$\boxed{e^A = e^a(I+B)}$$

QED

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If A is a 2×2 matrix, there is a matrix P (with "generalized" eigenvectors as columns) such that:

$$A = P B P^{-1},$$

where B is one of the following (see section 1.8):

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}; \quad B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}; \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then, the exponentials are

$$e^{Bt} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix}; \quad e^{Bt} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix};$$

$$\text{or} \quad e^{Bt} = e^{a}\begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

Therefore:

$$e^{At} = P e^{Bt} P^{-1},$$

for any of cases shown above.

Section 1.4. The fundamental Theorem of Linear Systems of Differential Equations.

Theorem.

Let A be an $n \times n$ matrix.

The Initial value problem:

$$\dot{x} = Ax$$

$$x(0) = x_0,$$

for $x_0 \in \mathbb{R}^n$, has a unique solution.

$$x(t) = e^{At} x_0.$$

Notice the similarity of the equation $\dot{x} = ax$, $x(t) = e^{at} x_0$, in 1-dim.

The proof is supported by the following Lemma:

Lemma: Let A be $n \times n$ matrix. Then

$$\frac{d}{dt} e^{At} = A e^{At}.$$

Proof Lemma:

$$\frac{d}{dt} e^{At} = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h} =$$

$$= \lim_{h \to 0} e^{At} \cdot \frac{e^{Ah} - I}{h}$$

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$$= e^{At} \lim_{h\to 0} \lim_{k\to\infty} \left(A + \frac{A^2 h}{2!_0} + \cdots + \frac{A^k h^{k-1}}{k!_0}\right)$$

Since the series. e^{Ah} converges uniformly for $|h| \leq h_0$,

then, we can interchange the limits

$$= e^{At} \lim_{k\to 0} \lim_{k\to 0} \left(A + \frac{A^2 h}{2!_0} + \cdots + \frac{A^k h^{k-1}}{k!_0}\right)$$

$$= e^{At} \lim_{k\to 0} A = e^{At} \cdot A.$$

$$= A e^{At}$$ Q.E.D. Lemma.

Proof of Thm: Since $x(t) \equiv e^{At} x_0$, then.

$$\dot{x}(t) = (e^{At})^{\cdot} x_0 = A e^{At} x_0 = A x(t).$$

Uniqueness is confusing in text. Assume two solutions $x(t)$, $y(t)$ to the initial value problem $\dot{x} = Ax$, $x(0) = x_0$.

I.e. $\dot{y} = Ay$, $y(0) = x_0$.

Then $(y-x)^{\cdot} = A(y-x)$. But $y(0) - x(0) = x_0 - x_0 = 0$.

We can check that $y - x = e^{At}\vec{c}$, satisfies the above equation, with $\bar{c}$ a constant vector. This vector should be:

$\vec{c} = y(0) - x(0) = 0$. Then: $y(t) - x(t) = 0$, and the soln. is unique QED.

Example: Solve: $\dot{x} = Ax$, $x(0) = x_0$.

with $A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$, $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

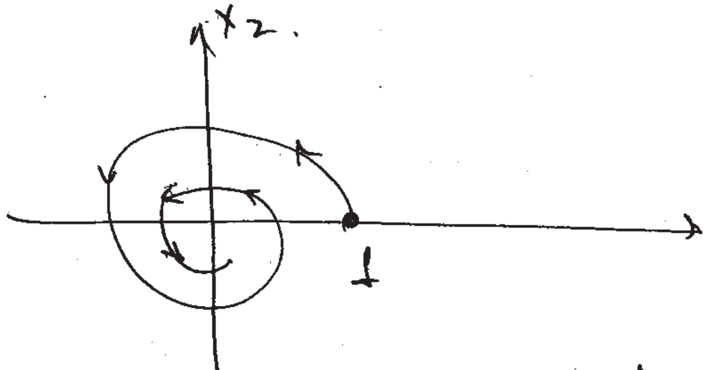
and sketch the solution in the phase-plane.

Notice $A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Then $$e^{At} = e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} = e^{-2t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and
$$x(t) = e^{-2t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boxed{x(t) = e^{-2t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}$$



Notice
$$\|x(t)\| = e^{-2t}$$
and $\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right)$
ie., $\theta(t) = \arctan\left(\frac{\sin t}{\cos t}\right)$
ie., $\boxed{\theta(t) = t.}$

The solution curve spirals into the origin.

1.5 Linear Systems in $\mathbb{R}^2$:

Here, we describe the phase portrait of

$$\dot{x} = Ax, \qquad (1)$$

for A a 2x2 matrix. We describe basically the trajectories described by:

$$\dot{x} = Bx, \qquad (2)$$

with $A = PBP^{-1}$, and this will be enough, since the solutions of this system can be taken into the former by $x \longrightarrow Px$.

The three possibilities are:

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}; \quad B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}; \text{ or } B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with: $x(t) = e^{Bt}x(0)$, then:

$$x(t) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} x(0); \quad x(t) = e^{\lambda t}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} x(0), \text{ and}$$

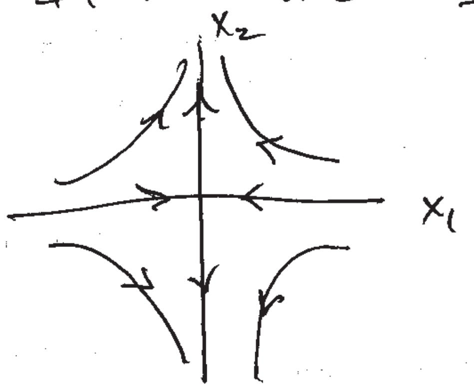
$$x(t) = e^{at}\begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} x(0),$$

solutions to system (2)

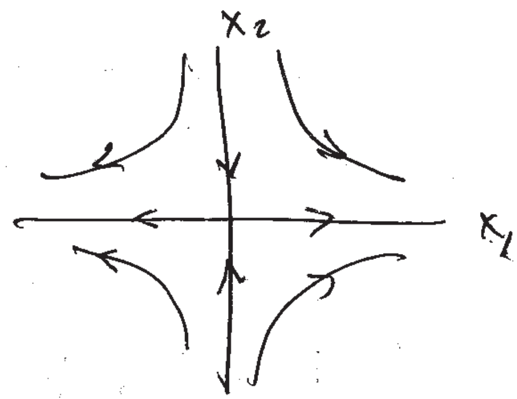
Case I $\quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda < 0 < \mu.$

This is an unstable equilibrium point. (the origin.)

It is named a saddle.



$\lambda < 0 < \mu$

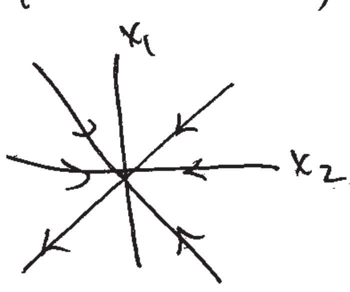


$\mu < 0 < \lambda$

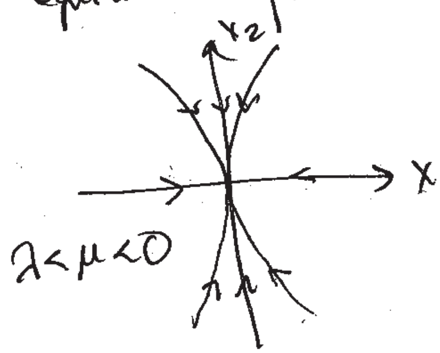
Case II $\quad B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ or $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

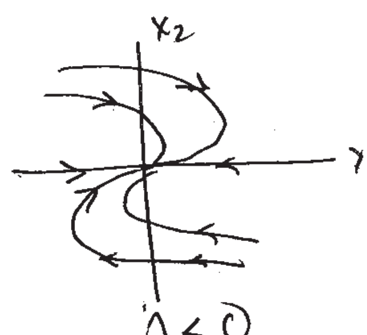
$\lambda \le \mu < 0. \qquad \lambda < 0.$

In this case, the equilibrium point is a stable node.



$\lambda = \mu < 0$





$\lambda < 0$

Equilibrium point is: Proper node $\qquad$ Improper node $\qquad$ Improper node

$\lambda \le \mu < 0$

$\lambda < 0$

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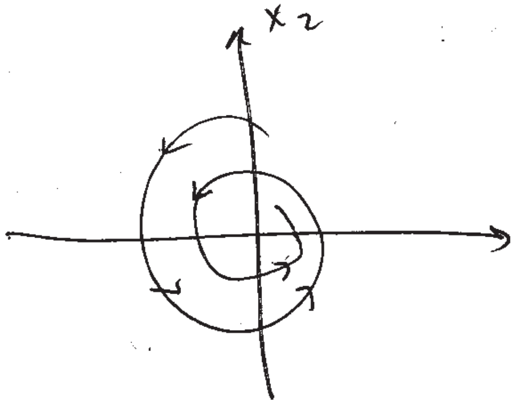
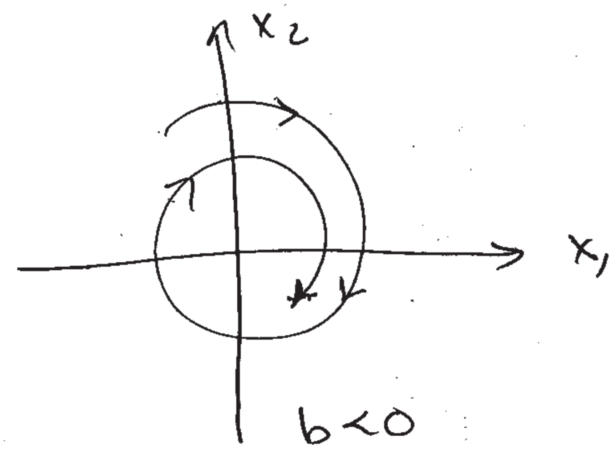
If $\lambda \geq \mu > 0$, and $\lambda > 0$,
the arrows are reversed, and we obtain unstable node.

Note that the curves approach to the origin (the fixed point) into a specific direction, θ_0, i.e.

$\theta(t) \xrightarrow[t \to \pm\infty]{} \theta$ ("+" if we have a stable node; "−" if we have an unstable node)

Case III $\quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$; $a < 0$.





These are called stable focus and they spiral in.
If $a > 0$, the direction of the arrows is reversed, and they are unstable focus.
This corresponds to the matrix with eivalues $\lambda = a \pm ib$.

This is to say, if the matrix A has eivals:

$$\lambda = a \pm ib.$$

then, the matrix $A = \begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix}$ can be taken into

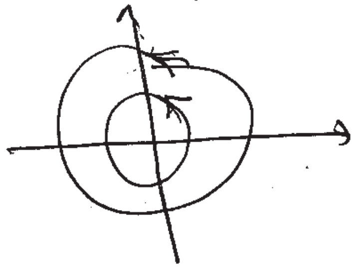
the form $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Notice that the trajectories do <u>not</u> approach the fixed point in a specified direction. Instead, they spiral into or out from the fixed point: $|\theta(t)| \xrightarrow[t \to \pm\infty]{} \infty$.

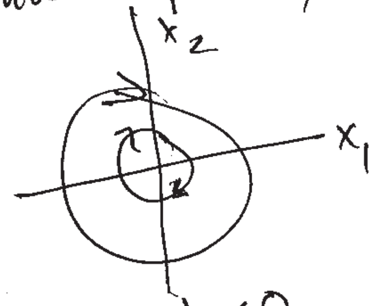
$\left(\text{again} \begin{cases} \text{"+"} & \text{stable focus} \\ \text{"−"} & \text{unstable focus} \end{cases} \right)$.

<u>Case IV</u> $B = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$

Here, we don't have spirals, but perfect circles:



$b > 0$



$b < 0$

Notice:

$|\theta(t)| \xrightarrow[t \to \infty]{} \infty$.

The eigenvalues of A are $\pm ib$.

We call this fixed point to be a <u>center</u>.

<u>Degenerate cases</u> If one or both of the eivalues of A is zero, $\det A = 0$, then, the fixed point is a <u>degenerate equilibrium point</u>.

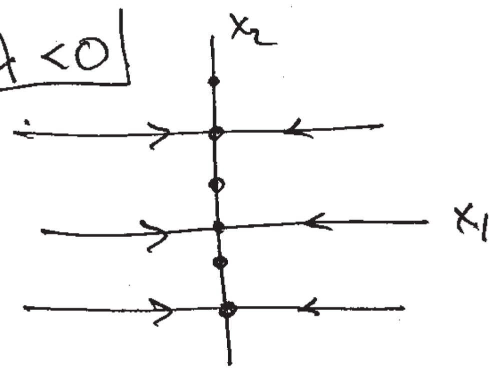
These cases are described in problem 4.

(a) $\dot{x} = Ax$, $\quad A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$

$$\left.\begin{aligned} \dot{x}_1 &= \lambda x_1 \\ \dot{x}_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1(t) &= x_1(0)\, e^{\lambda t} \\ x_2(t) &= \text{const} \end{aligned}$$

Motion along horizontal lines.

$\boxed{\lambda < 0}$



If $x_1(0) = 0$, we have a fixed point.

Then, all the points in the x_2-axis are fixed points.

Arrows are reversed if $\lambda > 0$.

(b) $\dot{x} = Ax$ $\quad A = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$

$$\left.\begin{aligned} \dot{x}_1 &= \gamma x_2 \\ \dot{x}_2 &= 0 \end{aligned}\right\} \Rightarrow x_2(t) = x_2(0)$$

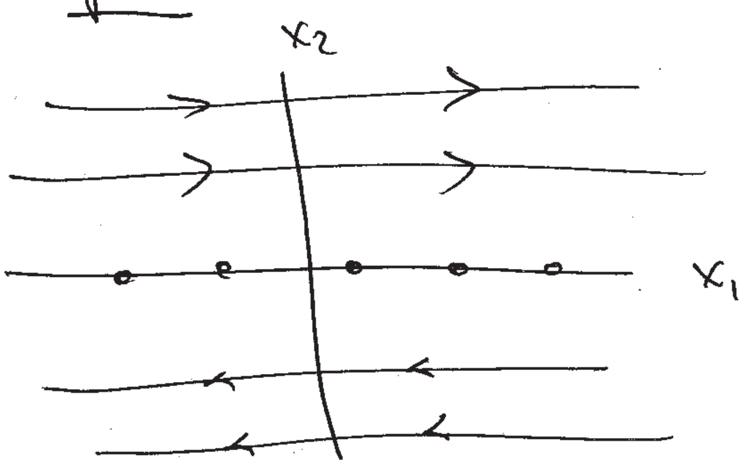
Motion along horizontal lines.

$\dot{x}_1 = \gamma\, x_2(0) \Rightarrow x_1(t) = \gamma\, x_2(0)\, t$

Straight lines $\boxed{x_1(t) = \gamma\, x_2(0)\, t + x_1(0)}$

If $x_2(0)=0$, Then, $x_1(t)=x_1(0)$ is a constant.

Then, any point in the x_1-axis is a fixed point



Assume $\gamma>0$.

* If $x_2(0)>0$, $\Rightarrow \dot{x}_1>0$.
  then motion to the right, from $-\infty$ to $+\infty$
* If $x_2(0)<0 \Rightarrow \dot{x}_2<0$
  then motion to the left from $-\infty$ to $+\infty$.

If $\gamma<0$, the arrows reverse direction

Example of a center.

Solve $\dot{x}=Ax$ and plot is phase-portrait,

with $A=\begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$. ; $A-\lambda I=\begin{pmatrix} -\lambda & -4 \\ 1 & -\lambda \end{pmatrix}$.

Eivals:

$\lambda^2+4=0$

$\boxed{\lambda_{1,2}=\pm 2i.}$

$\lambda_1=-2i$

$\lambda_2=2i$

Eivectors.

$w_1=\begin{pmatrix} 2 \\ i \end{pmatrix}$; $w_2=\begin{pmatrix} 2 \\ -i \end{pmatrix}$

Then $w_1=u_1+iv_2=\begin{pmatrix} 2 \\ 0 \end{pmatrix}+i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $A=\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ +2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

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(c) If $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1(t) &= x_1(0) \\ x_2(t) &= x_2(0) \end{aligned}$$ are all constants.

Then, any point is a <u>fixed point</u>

Then: $A = P B P^{-1}$.

$$e^{At} = P\, e^{Bt} P^{-1} = P\, e^{0t}\begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} P^{-1}$$

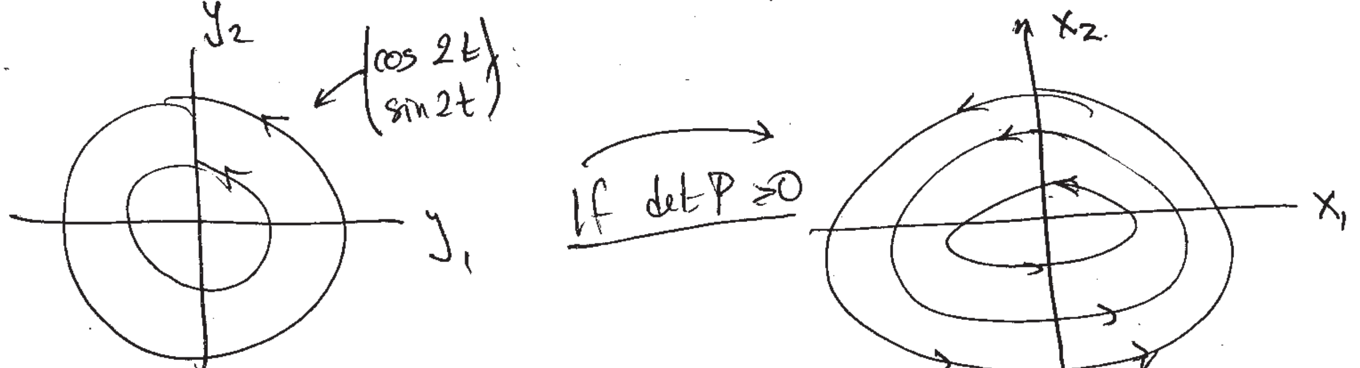
$$= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 2\cos 2t & -2\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 2\cos 2t & -2\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} \cos 2t & -2\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{pmatrix}$$

The solution is: $\vec{x}(t) = e^{At}\vec{x}(0) = \vec{x}(t) = \begin{pmatrix} x_1(0)\cos 2t - 2x_2(0)\sin 2t \\ \frac{x_1(0)}{2}\sin 2t + x_2(0)\cos 2t \end{pmatrix}$



You can verify that $x_1^2(t) + x_2^2(t) = c_1^2 + 4c_2^2$: Ellipse

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Remark: Notice we considered the change of variables:

$$y = P^{-1} x$$

If P is orientation preserving, i.e.,

$$\det P \geq 0,$$

then, the orientation of the curves is the same either on the y-plane or the x-plane.

On the other hand, if P is orientation reversing, this is,

$$\det < 0,$$

then, the trajectory solutions are reversed in orientation from the plane y-plane to the x-plane

– o —

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The following Theorem helps us to determine the nature of the fixed point.

We require $\det A \neq 0$. This implies that if x_0 is a fixed point, $\dot{x}_0 = 0$, then $Ax_0 = 0$, and so, $x_0(t) \equiv 0$, is the only fixed point for a linear system.

Now, if we compute: $\dot{x}_2$ at $(0, x_2)$, we can determine if the motion is clockwise ($\dot{x}_2 > 0$) or counter-clockwise ($\dot{x}_2 < 0$) for a focus or a center.

We know have the following theorem.

Theorem: Let: $\delta = \det A$
Given A (2×2 matrix) $\det A \neq 0$. $\tau = \text{trace } A$

Consider the linear system: $\dot{x} = Ax$. Then:

(a) If $\delta < 0$, then fixed point is a saddle.

(b) If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ } fixed point <u>node</u> = { stable : $\tau < 0$; unstable $\tau > 0$.

(c) If $\delta > 0$, and $\tau^2 - 4\delta < 0$ } fixed point is a <u>focus</u> = { stable : $\tau < 0$; unstable: $\tau > 0$

(d) If $\delta > 0$, and $\tau = 0$, fixed point is a <u>center</u>: <u>Neutrally</u> stable.

Note 1: In case (b):

$\tau^2 \geq 4\delta > 0$, then $\tau \neq 0$ automatically.

Note 2: In case (c) $\tau^2 - 4\delta < 0$, and $\tau > 0$ or $\tau < 0$. The intermediate case is: $\tau = 0$, then $0^2 - 4\delta < 0 \Rightarrow \delta > 0$, hence $\delta > 0$, and $\tau = 0$ is case (d). I.e., case (d) follows from case (c).

We need to remember that for:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with eigenvalues λ_1, λ_2:

$$\delta = \det A = ad - cb = \lambda_1 \lambda_2$$

$$\tau = \operatorname{tr} A = a + d = \lambda_1 + \lambda_2.$$

Proof: (a) If $\delta < 0$, the two eigenvals have opposite sign, then saddle.

(b) If $\delta > 0$, the two eivals have same sign.

If $\tau^2 - 4\delta \geq 0$, then, two eigenvalues are real, since

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$ are the eivals of A.

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If $\tau < 0$, both eivals are < 0,

If $\tau > 0$, then both eivals are > 0,

by virtue of $\tau = \lambda_1 + \lambda_2$, and $\lambda_1 \lambda_2 = \delta > 0$,

they have the same sign.

Then, if $\tau < 0$, $\underbrace{\text{it}}_{\text{fixed point}}$ is stable node

and if $\tau < 0$, fixed point is unstable node.

(c) If $\delta > 0$, and $\tau^2 - 4\delta < 0$, $\lambda_1 = \lambda_2^*$ are complex. If $\tau < 0$, the fixed point stable. If $\tau > 0$, fixed point unstable.

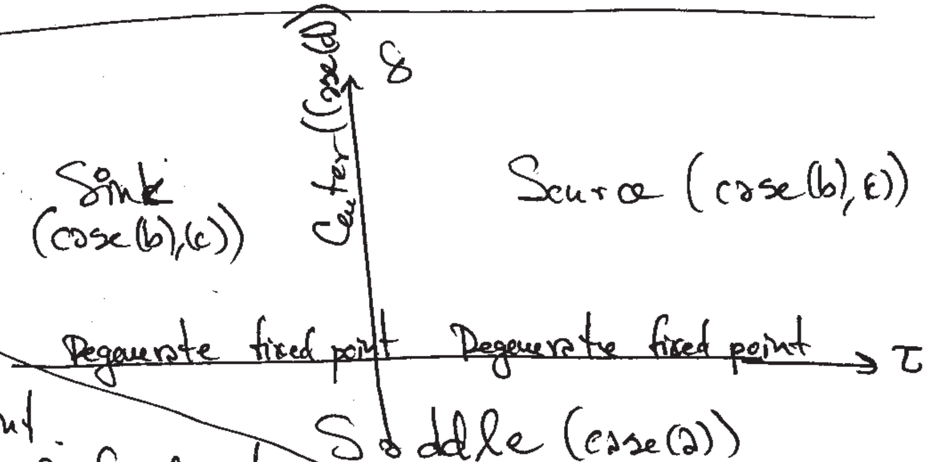
focus $\Big\{ \lambda_{1,2} = \frac{\tau}{2} \pm i \frac{\sqrt{4\delta - \tau^2}}{2}$

(d) If $\delta > 0$ and $\tau = 0$, $\Rightarrow$ τ $\lambda_{1,2} = \frac{\pm i 2\sqrt{\delta}}{2}$

$\Rightarrow \lambda_{1,2} = \pm i\sqrt{\delta}$

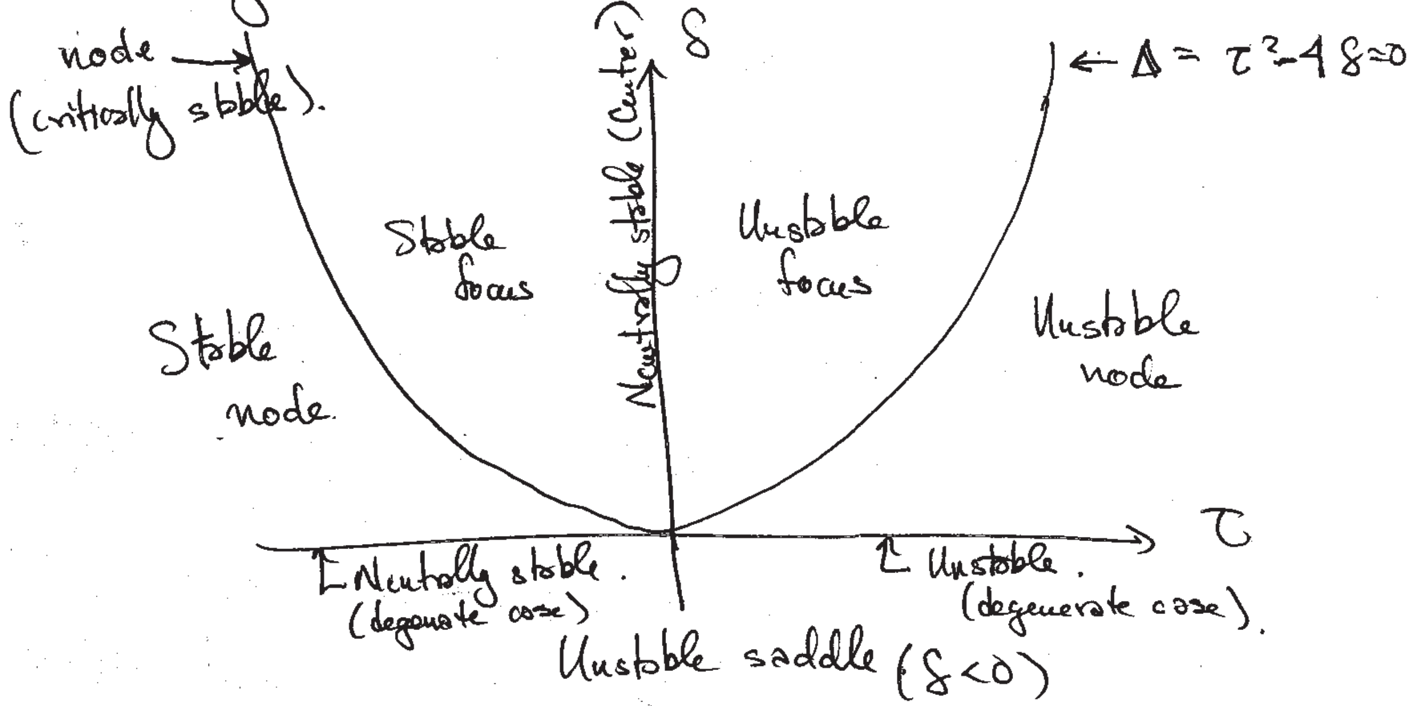
$\Rightarrow$ we have a center.

Bifurcation diagram



The sink is the stable fixed point.
All the rest are unstable fixed points.

According to Richard Haberman's book (Mathematical Models)



According to Perko, there are 8 different possibilities for the fixed point: Cases I, II, III, IV, plus 4 cases from the degenerate case $\delta = 0$ (see Problem 1.4).

END NOTES FROM:

Perko. *Differential Equations and Dynamical Systems*.