A TASTE of QUANTUM MECHANICS | Friday, March 19, 2010

Derivation of the dispersion relation for "matter waves".

(Taken from:
Applied Asymptotic Analysis.
by Peter Miller AMS 2006)
(Graduate Studies in Mathematics Vol 75).

The derivation of the Schrödinger equation became differently from other Mathematical Physics phenomena. In this instance, experimental evidence of "matter waves" was reported in the literature.

There were no works trying to explain what a "matter wave" was, but explain its physical-mechanical properties instead. This is to say, experiments showed ~~what~~ the relation among the physical properties of the "matter wave" such as frequency and wave number, to their mechanical properties, such as momentum and energy. This is to say, we wanto to know the dispersion relation of the "matter wave" (whatever it is).

In mechanics, in Classical Mechanics, there is a relation between the total energy, and parts of it, namely

$$E = T + V$$

is the total energy, T = kinetic energy of a particle
V = potential energy of a particle that comes from external forces

In the experiments, we have free particles so that $V \equiv 0$ and:

$$E = T$$

i.e.

$$E = \frac{p^2}{2m}.$$

The dispersion relation will come from two hypothesis from Quantum Mechanics, as follows.

<u>First hypothesis:</u>

To start with, remember that one of the main facts in ondulatory mechanics is that:

$$\lambda k = 2\pi, \quad \text{where} \quad \begin{array}{l} \lambda = \text{wave length} \\ k = \text{wave number} \end{array}$$

Now, experiments have shown that the momentum of the particles, p, is related to its wave length:

$$p = \frac{h}{\lambda},$$

where "h" is the Plack's Constant: (Planck)

$$h = 6.626068 \times 10^{-34} [m^2 kg/seg] [J \cdot seg].$$

or

$$h = 6.626068 \times 10^{-27} \text{ erg} \cdot \text{seg}$$

Very small!

Using eq. (1) into (2), we get:

$$\boxed{p = \hbar k}$$ ← Fist hypothesis.

which is the DeBroglie relation. Here:

$$\hbar \equiv \frac{h}{2\pi} = 1.054571 \times 10^{-34}\,\mathrm{J \cdot seg.}$$

$$= 1.054871 \times 10^{-27}\,\mathrm{erg \cdot seg.}$$

still very small.

Here k and λ are associated to the wavenumber and wavelength of the "matter wave".

<u>Second hypothesis</u> <u>of</u> <u>Quantum</u> <u>Mechanics</u>.

The energy, the total energy E of a particle is associated to the frequency "f" carried by the "matter wave":

$$E = h \cdot f.$$

or:

$$\boxed{E = \hbar \Omega,}$$ ← Second hypothesis

where $\Omega \equiv 2\pi f$ is the angular frequency associated to the "matter wave" with frequency "f".

This is the famous Planck's statement that the energy can only be emitted in "discrete packages" called "quanta": this is to say, given a frequency Ω, the energy can be emitted as multiples of $E = \hbar\Omega$.

Combine now the two hypothesis.

We now combine these two hypothesis into the relation

$$E = \frac{p^2}{2m} \; , \quad \cdots\cdots \quad (3)$$

The total energy $E = \hbar\Omega$ should be equal to the kinetic energy $T = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$, when the two fields are in resonance; i.e., the the frequency of the an external field, Ω, (with energy $E = \hbar\Omega$ in the form of a photon, to put into motion the particle) should be equal to the frequency ω (of the particle in motion with momentum: $p = \hbar k$):

$$\Omega = \omega .$$

i.e.

$$E = \hbar\omega .$$

Then, eq. (3):

$$E = \frac{p^2}{2m} ,$$ and using both Q.M. hypothesis

we get:

$$\hbar\omega = \frac{(\hbar k)^2}{2m}$$

$$\boxed{\hbar\omega = \frac{\hbar^2 k^2}{2m}} \quad \cdots\cdots \quad (4).$$

which is the dispersion relation for "matter waves".

The linear partial differential equation for free "matter waves":

The dispersion relation (4) comes out once we substitute the ansatz of solutions for linear waves:

$$\psi = e^{i(kx - \omega t)},$$

into the equation:

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi$$

This is the Schrödinger equation for a free particle.
Now, what Schrödinger postulated, (as an analogy to the conservation ~~law~~ of energy from Classical Mechanics) was the equation

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi.}$$

which is the famous Schrödinger equation of Quantum Mechanics.

Afterwards, this equation was tested on many experiments being a successful model for the quantum world.