3. *ω consists of equilibria of (17.0.2) that are cyclically connected to each other in a* heteroclinic cycle. *In the case of only one equilibrium it would be connected by a homoclinic orbit.*

Proof: See Thieme [1992]. □

Further information on asymptotically autonomous systems can be found in Thieme [1994], as well as the original work of Markus [1956]. Thieme [1994] contains a number of examples that appear to be somewhat counter-intuitive. Holmes and Stuart [1992] study the existence of homoclinic orbits in asymptotically autonomous vector fields. Asymptotically autonomous equations also naturally arise in the study of the approach of trajectories to a low dimensional invariant manifold in autonomous systems, see Robinson [1996].

17.1 Exercises

1. Consider the following asymptotically autonomous vector field in the plane:

$$\dot{x} = y,$$
$$\dot{y} = x - x^3 - \delta y + \gamma e^{-t}, \qquad (x, y) \in \mathbb{R}^2, \quad \delta, \gamma > 0.$$

Describe the ω limit sets for trajectories.

2. This example is from Thieme [1994]. Consider the following planar vector field

$$\begin{aligned} \dot{x} &= (-x(1-x) + y)(2+x), \\ \dot{y} &= -y. \end{aligned} \qquad (17.1.1)$$

The y component of the vector field is independent of the x component and the solution of this can be solved and substituted into the x component to yield the following asymptotically autonomous system

$$\dot{x} = \left(-x(1-x) + y_0 e^{-t}\right)(2+x), \qquad (17.1.2)$$

and the asymptotically autonomous limit of this equation is

$$\dot{x} = -x(1-x)(2+x). \qquad (17.1.3)$$

(a) Show that $x = 0$ is an attracting equilibrium of (17.1.3), and that the basin of attraction is the open interval between $x = 1$ and $x = -2$.

(b) For (17.1.2), show that the basin of attraction of $x = 0$ becomes arbitrarily small as y_0 is chosen arbitrarily large.

18

Center Manifolds

When one thinks of simplifying dynamical systems, two approaches come to mind: one, reduce the dimensionality of the system and two, eliminate the nonlinearity. Two rigorous mathematical techniques that allow substantial progress along both lines of approach are center manifold theory and the method of normal forms. These techniques are the most important, generally applicable methods available in the local theory of dynamical systems, and they will form the foundation of our development of bifurcation theory in Chapters 20 and 21.

The center manifold theorem in finite dimensions can be traced to the work of Pliss [1964], Šošitaĭšvili [1975], and Kelley [1967]. Additional valuable references are Guckenheimer and Holmes [1983], Hassard, Kazarinoff, and Wan [1980], Marsden and McCracken [1976], Carr [1981], Henry [1981], and Sijbrand [1985].

The method of normal forms can be traced to the Ph.D thesis of Poincaré [1929]. The books by van der Meer [1985] and Bryuno [1989] give valuable historical background.

Let us begin our discussion of center manifold theory with some motivation. Consider the linear systems

$$\dot{x} = Ax, \qquad (18.0.1)$$
$$x \longmapsto Ax, \qquad x \in \mathbb{R}^n, \qquad (18.0.2)$$

where A is an $n \times n$ matrix. Recall from Chapter 3 that each system has invariant subspaces E^s, E^u, E^c, corresponding to the span of the generalized eigenvectors, which in turn correspond to eigenvalues having

Flows: negative real part, positive real part, and zero real part, respectively.

Maps: modulus < 1, modulus > 1, and modulus $= 1$, respectively.

The subspaces were so named because orbits starting in E^s decayed to zero as t (resp. n for maps) $\uparrow \infty$, orbits starting in E^u became unbounded as t (resp. n for maps) $\uparrow \infty$, and orbits starting in E^c neither grew nor decayed *exponentially* as t (resp. n for maps) $\uparrow \infty$.

If we suppose that $E^u = \emptyset$, then we find that any orbit will rapidly decay to E^c. Thus, if we are interested in long-time behavior (i.e., stability) we need only to investigate the system restricted to E^c.

It would be nice if a similar type of "reduction principle" applied to the study of the stability of nonhyperbolic fixed points of nonlinear vector fields and maps, namely, that there were an invariant *center manifold* passing through the fixed point to which the system could be restricted in order to study its asymptotic behavior in the neighborhood of the fixed point. That this is the case is the content of the center manifold theory.

18.1 Center Manifolds for Vector Fields

We will begin by considering center manifolds for vector fields. The set-up is as follows. We consider vector fields of the following form

$$\begin{aligned} \dot{x} &= Ax + f(x, y), \\ \dot{y} &= By + g(x, y), \qquad (x, y) \in \mathbb{R}^c \times \mathbb{R}^s, \end{aligned} \tag{18.1.1}$$

where

$$\begin{aligned} f(0,0) &= 0, \quad Df(0,0) = 0, \\ g(0,0) &= 0, \quad Dg(0,0) = 0. \end{aligned} \tag{18.1.2}$$

(See Chapter 3 for a discussion of how a general vector field is transformed to the form of (18.1.1) in the neighborhood of a fixed point.)

In the above, A is a $c \times c$ matrix having eigenvalues with zero real parts, B is an $s \times s$ matrix having eigenvalues with negative real parts, and f and g are $\mathbf{C}^r$ functions ($r \geq 2$).

Definition 18.1.1 (Center Manifold) *An invariant manifold will be called a center manifold for (18.1.1) if it can locally be represented as follows*

$$W^c(0) = \{ (x, y) \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0 \}$$

for δ sufficiently small.

We remark that the conditions $h(0) = 0$ and $Dh(0) = 0$ imply that $W^c(0)$ is tangent to E^c at $(x, y) = (0, 0)$. The following three theorems are taken from the excellent book by Carr [1981].

The first result on center manifolds is an existence theorem.

Theorem 18.1.2 (Existence) *There exists a $\mathbf{C}^r$ center manifold for (18.1.1). The dynamics of (18.1.1) restricted to the center manifold is, for u sufficiently small, given by the following c-dimensional vector field*

$$\dot{u} = Au + f(u, h(u)), \qquad u \in \mathbb{R}^c. \tag{18.1.3}$$

Proof: See Carr [1981]. □

The "u" Notation. Since the center manifold of an equilibrium point is locally represented as a graph, i.e., $y = h(x)$, the reader may be wondering why we substituted u for x in the restriction of the vector field to the center manifold given in (18.1.3). This is to emphasize that the restriction of the vector field to the center manifold is, generally, a vector field on a *nonlinear surface*. If we had used x, since $(x, y) \in \mathbb{R}^c \times \mathbb{R}^s$ are the original coordinates for the vector field, this point might have been be obscured. Once this point of interpretation is understood, there is no harm in using x (or, for that matter, any other symbol), and this is typically done in the literature.

The next result implies that the dynamics of (18.1.3) near $u = 0$ determine the dynamics of (18.1.1) near $(x, y) = (0, 0)$.

Theorem 18.1.3 (Stability) i) *Suppose the zero solution of (18.1.3) is stable (asymptotically stable) (unstable); then the zero solution of (18.1.1) is also stable (asymptotically stable) (unstable).* ii) *Suppose the zero solution of (18.1.3) is stable. Then if $(x(t), y(t))$ is a solution of (18.1.1) with $(x(0), y(0))$ sufficiently small, there is a solution $u(t)$ of (18.1.3) such that as $t \to \infty$*

$$\begin{aligned} x(t) &= u(t) + \mathcal{O}(e^{-\gamma t}), \\ y(t) &= h(u(t)) + \mathcal{O}(e^{-\gamma t}), \end{aligned}$$

where $\gamma > 0$ is a constant.

Proof: See Carr [1981]. □

Dynamics Captured by the Center Manifold

Stated in words, this theorem says that for initial conditions of the *full system* sufficiently close to the origin, trajectories through them asymptotically approach a trajectory on the center manifold. In particular, equilibrium points sufficiently close to the origin, sufficiently small amplitude periodic orbits, as well as "small" homoclinic and heteroclinic orbits *are contained in the center manifold.*

The obvious question now is how do we compute the center manifold so that we can reap the benefits of Theorem 18.1.3? To answer this question we will derive an equation that $h(x)$ must satisfy in order for its graph to be a center manifold for (18.1.1).

Suppose we have a center manifold

$$W^c(0) = \{(x, y) \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x), |x| < \delta, h(0) = 0, \ Dh(0) = 0\}, \tag{18.1.4}$$

with δ sufficiently small. Using invariance of $W^c(0)$ under the dynamics of (18.1.1), we derive a quasilinear partial differential equation that $h(x)$ must satisfy. This is done as follows:

1. The (x, y) coordinates of any point on $W^c(0)$ must satisfy
$$y = h(x). \tag{18.1.5}$$

2. Differentiating (18.1.5) with respect to time implies that the $(\dot{x}, \dot{y})$ coordinates of any point on $W^c(0)$ must satisfy
$$\dot{y} = Dh(x)\dot{x}. \tag{18.1.6}$$

3. Any point on $W^c(0)$ obeys the dynamics generated by (18.1.1). Therefore, substituting
$$\dot{x} = Ax + f\big(x, h(x)\big), \tag{18.1.7}$$
$$\dot{y} = Bh(x) + g\big(x, h(x)\big) \tag{18.1.8}$$
into (18.1.6) gives
$$Dh(x)\big[Ax + f\big(x, h(x)\big)\big] = Bh(x) + g\big(x, h(x)\big) \tag{18.1.9}$$
or
$$\mathcal{N}\big(h(x)\big) \equiv Dh(x)\big[Ax + f\big(x, h(x)\big)\big] - Bh(x) - g\big(x, h(x)\big) = 0. \tag{18.1.10}$$

Equation (18.1.10) is a quasilinear partial differential equation that $h(x)$ must satisfy in order for its graph to be an invariant center manifold. To find a center manifold, all we need do is solve (18.1.10).

Unfortunately, it is probably more difficult to solve (18.1.10) than our original problem; however, the following theorem gives us a method for computing an approximate solution of (18.1.10) to any desired degree of accuracy.

Theorem 18.1.4 (Approximation) *Let $\phi : \mathbb{R}^c \to \mathbb{R}^s$ be a $\mathbf{C}^1$ mapping with $\phi(0) = D\phi(0) = 0$ such that $\mathcal{N}\big(\phi(x)\big) = \mathcal{O}(|x|^q)$ as $x \to 0$ for some $q > 1$. Then*
$$|h(x) - \phi(x)| = \mathcal{O}(|x|^q) \qquad \text{as } x \to 0.$$

Proof: See Carr [1981]. □

This theorem allows us to compute the center manifold to any desired degree of accuracy by solving (18.1.10) to the same degree of accuracy. For this task, power series expansions will work nicely. Let us consider a concrete example.

Example 18.1.1. Consider the vector field
$$\begin{aligned} \dot{x} &= x^2 y - x^5, \\ \dot{y} &= -y + x^2, \qquad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{18.1.11}$$

The origin is obviously a fixed point for (18.1.11), and the question we ask is whether or not it is stable. The eigenvalues of (18.1.11) linearized about $(x, y) = (0, 0)$ are 0 and -1. Thus, since the fixed point is not hyperbolic, we cannot make any conclusions concerning the stability or instability of $(x, y) = (0, 0)$ based on linearization (note: in the linear approximation the origin is stable but not asymptotically stable). We will answer the question of stability using center manifold theory.

From Theorem 18.1.2, there exists a center manifold for (18.1.11) which can locally be represented as follows
$$W^c(0) = \big\{ (x, y) \in \mathbb{R}^2 \mid y = h(x), |x| < \delta,\ h(0) = Dh(0) = 0 \big\} \tag{18.1.12}$$
for δ sufficiently small. We now want to compute $W^c(0)$. We assume that $h(x)$ has the form
$$h(x) = ax^2 + bx^3 + \mathcal{O}(x^4), \tag{18.1.13}$$
and we substitute (18.1.13) into equation (18.1.10), which $h(x)$ must satisfy to be a center manifold. We then equate equal powers of x, and in that way we can compute $h(x)$ to any desired order of accuracy. In practice, computing only a few terms is usually sufficient to answer questions of stability.

We recall from (18.1.10) that the equation for the center manifold is given by
$$\mathcal{N}(h(x)) = Dh(x)\left[Ax + f(x, h(x))\right] - Bh(x) - g(x, h(x)) = 0, \tag{18.1.14}$$
where, in this example, we have $(x, y) \in \mathbb{R}^2$,
$$\begin{aligned} A &= 0, \\ B &= -1, \\ f(x, y) &= x^2 y - x^5, \\ g(x, y) &= x^2. \end{aligned} \tag{18.1.15}$$

Substituting (18.1.13) into (18.1.14) and using (18.1.15) gives
$$\begin{aligned} \mathcal{N}(h(x)) = {} & (2ax + 3bx^2 + \cdots)(ax^4 + bx^5 - x^5 + \cdots) \\ & + ax^2 + bx^3 - x^2 + \cdots = 0. \end{aligned} \tag{18.1.16}$$

In order for (18.1.16) to hold, the coefficients of each power of x must be zero; see Exercise 2. Thus, equating coefficients on each power of x to zero gives
$$\begin{aligned} x^2 &: a - 1 = 0 \Rightarrow a = 1, \\ x^3 &: b = 0, \\ &\vdots \quad \vdots \end{aligned} \tag{18.1.17}$$
and we therefore have
$$h(x) = x^2 + \mathcal{O}(x^4). \tag{18.1.18}$$

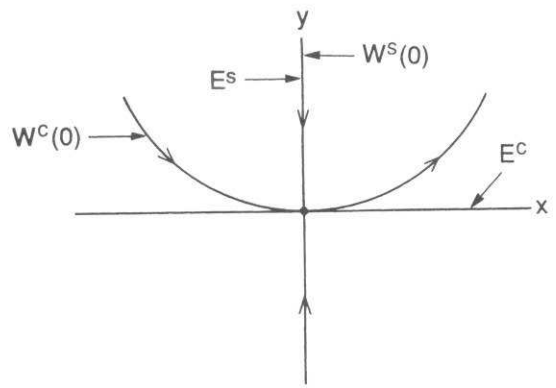

FIGURE 18.1.1.

Using (18.1.18) along with Theorem 18.1.2, the vector field restricted to the center manifold is given by

$$\dot{x} = x^4 + \mathcal{O}(x^5). \qquad (18.1.19)$$

For x sufficiently small, $x = 0$ is thus unstable in (2.1.19). Hence, by Theorem 18.1.2, $(x, y) = (0, 0)$ is unstable in (18.1.11); see Figure 18.1.1 for an illustration of the geometry of the flow near $(x, y) = (0, 0)$.

This example illustrates an important phenomenon, which we now describe.

The Failure of the Tangent Space Approximation

The idea is as follows. Consider (18.1.11). One might expect that the y components of orbits starting near $(x, y) = (0, 0)$ should decay to zero exponentially fast. Therefore, the question of stability of the origin should reduce to a study of the x component of orbits starting near the origin. One might thus be very tempted to set $y = 0$ in (18.1.11) and study the reduced equation

$$\dot{x} = -x^5. \qquad (18.1.20)$$

This corresponds to approximating $W^c(0)$ by E^c. However, $x = 0$ is stable for (18.1.20) and, therefore, we would arrive at the *wrong* conclusion that $(x, y) = (0, 0)$ is stable for (18.1.20). The tangent space approximation might sometimes work, but, as this example shows, it does not always do so.

End of Example 18.1.1

Example 18.1.2. The previous example showed a situation where an equilibrium point was unstable, but the tangent space approximation to its center manifold indicated that it was stable. One could ask the following question. "Suppose the equilibrium point is stable, will the tangent space approximation to the center manifold also show stability"? Here we give an example showing that the answer is "no".

Consider the vector field

$$\begin{aligned}\dot{x} &= -xy - x^6,\\ \dot{y} &= -y + x^2, \qquad (x, y) \in \mathbb{R}^2.\end{aligned}$$

The origin is an equilibrium point, and the eigenvalues of the matrix associated with the linearization are 0 and -1. The tangent space to the center manifold is the x axis. Hence, the restriction of the vector field to the center manifold *in the tangent space approximation* is given by

$$\dot{x} = -x^6,$$

for which the origin is unstable.

The center manifold can be calculated, and it is given by the graph of the following function

$$h(x) = x^2 + \mathcal{O}(4).$$

The vector field restricted to the center manifold is given by

$$\dot{x} = -x^3 + \mathcal{O}(5),$$

which indicates that the origin is stable.

End of Example 18.1.2

18.2 Center Manifolds Depending on Parameters

Suppose (18.1.1) depends on a vector of parameters, say $\varepsilon \in \mathbb{R}^p$. In this case we write (2.1.2) in the form

$$\begin{aligned}\dot{x} &= Ax + f(x, y, \varepsilon),\\ \dot{y} &= By + g(x, y, \varepsilon), \qquad (x, y, \varepsilon) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p, \qquad (18.2.1)\end{aligned}$$

where

$$\begin{aligned}f(0, 0, 0) &= 0, \qquad Df(0, 0, 0) = 0,\\ g(0, 0, 0) &= 0, \qquad Dg(0, 0, 0) = 0,\end{aligned}$$

and we have the same assumptions on A and B as in (18.1.1), with f and g also being $\mathbf{C}^r$ $(r \geq 2)$ functions in some neighborhood of $(x, y, \varepsilon) = (0, 0, 0)$. An obvious question is why do we not allow the matrices A and B to depend on ε? This will be answered shortly.

The way in which we will handle parametrized systems is to include the parameter ε as a new *dependent variable* as follows

$$\begin{aligned}\dot{x} &= Ax + f(x, y, \varepsilon),\\ \dot{\varepsilon} &= 0,\\ \dot{y} &= By + g(x, y, \varepsilon), \qquad (x, \varepsilon, y) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^s. \qquad (18.2.2)\end{aligned}$$

At first glance it might appear that nothing is really gained from this action, but we will argue otherwise.

Let us suppose we are considering (18.2.2) afresh. It obviously has a fixed point at $(x, \varepsilon, y) = (0, 0, 0)$. The matrix associated with the linearization of (18.2.2) about this fixed point has $c + p$ eigenvalues with zero real part and s eigenvalues with negative real part. Now let us apply center manifold theory. Modifying Definition 18.1.1, a center manifold will be represented as a graph over the *x and ε variables*, i.e., the graph of $h(x, \varepsilon)$ for x and ε sufficiently small. Theorem 18.1.2 still applies, with the vector field reduced to the center manifold given by

$$\begin{aligned} \dot{u} &= Au + f(u, h(u, \varepsilon), \varepsilon), \\ \dot{\varepsilon} &= 0, \qquad (u, \varepsilon) \in \mathbb{R}^c \times \mathbb{R}^p. \end{aligned} \tag{18.2.3}$$

Theorems 18.1.3 and 18.1.4 also follow (we will worry about any modifications to computing the center manifold shortly). Thus, adding the parameter as a new dependent variable merely acts to augment the matrix A in (18.1.1) by adding p new center directions that have no dynamics, and the theory goes through just the same. However, there is a new concept which will be important when we study *bifurcation theory*; namely, the center manifold exists for all ε in a sufficiently small neighborhood of $\varepsilon = 0$. We will learn in Chapters 20 and 21 that it is possible for solutions to be created or destroyed by perturbing nonhyperbolic fixed points. Thus, since the invariant center manifold exists in a sufficiently small neighborhood in both x and ε of $(x, \varepsilon) = (0, 0)$, all bifurcating solutions will be contained in the lower dimensional center manifold.

Let us now worry about computing the center manifold. From the existence theorem for center manifolds, locally we have

$$\begin{aligned} W^c_{\text{loc}}(0) = \big\{ (x, \varepsilon, y) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^s \mid y = h(x, \varepsilon), |x| < \delta, \\ |\varepsilon| < \bar{\delta},\ h(0,0) = 0,\ Dh(0,0) = 0 \big\} \end{aligned} \tag{18.2.4}$$

for δ and $\bar{\delta}$ sufficiently small. Using invariance of the graph of $h(x, \varepsilon)$ under the dynamics generated by (18.2.2) we have

$$\dot{y} = D_x h(x, \varepsilon)\dot{x} + D_\varepsilon h(x, \varepsilon)\dot{\varepsilon} = Bh(x, \varepsilon) + g\left(x, h(x, \varepsilon), \varepsilon\right). \tag{18.2.5}$$

However,

$$\begin{aligned} \dot{x} &= Ax + f\left(x, h(x, \varepsilon), \varepsilon\right), \\ \dot{\varepsilon} &= 0; \end{aligned} \tag{18.2.6}$$

hence substituting (18.2.6) into (18.2.5) results in the following quasilinear partial differential equation that $h(x, \varepsilon)$ must satisfy in order for its graph to be a center manifold.

$$\begin{aligned} \mathcal{N}\left(h(x, \varepsilon)\right) = D_x h(x, \varepsilon) \left[Ax + f\left(x, h(x, \varepsilon), \varepsilon\right)\right] \\ - Bh(x, \varepsilon) - g\left(x, h(x, \varepsilon), \varepsilon\right) = 0. \end{aligned} \tag{18.2.7}$$

Thus, we see that (18.2.7) is very similar to (18.1.10).

Before considering a specific example we want to point out an important fact. By considering ε as a new dependent variable, terms such as

$$x_i \varepsilon_j, \qquad 1 \le i \le c, \quad 1 \le j \le p,$$

or

$$y_i \varepsilon_j, \qquad 1 \le i \le s, \quad 1 \le j \le p,$$

become *nonlinear terms*. In this case, returning to a question asked at the beginning of this section, the parts of the matrices A and B depending on ε are now viewed as nonlinear terms and are included in the f and g terms of (18.2.2), respectively. We remark that in applying center manifold theory to a given system, it must first be transformed into the standard form (either (18.1.1) or (18.2.2)).

Example 18.2.1 (The Lorenz Equations). Consider the Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \bar{\rho}x + x - y - xz, \qquad (x, y, z) \in \mathbb{R}^3, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{18.2.8}$$

where σ and β are viewed as fixed positive constants and $\bar{\rho}$ is a parameter (note: in the standard version of the Lorenz equations it is traditional to put $\bar{\rho} = \rho - 1$). It should be clear that $(x, y, z) = (0, 0, 0)$ is a fixed point of (18.2.9). Linearizing (18.2.9) about this fixed point, we obtain the associated matrix

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}. \tag{18.2.9}$$

(Note: recall, $\bar{\rho}x$ is a nonlinear term.)

Since (18.2.9) is in block form, the eigenvalues are particularly easy to compute and are given by

$$0, -\sigma - 1, -\beta, \tag{18.2.10}$$

with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{18.2.11}$$

Our goal is to determine the nature of the stability of $(x, y, z) = (0, 0, 0)$ for $\bar{\rho}$ near zero. First, we must put (18.2.9) into the standard form (18.2.2). Using the eigenbasis (18.2.11), we obtain the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{18.2.12}$$

with inverse

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{1+\sigma} \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1+\sigma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{18.2.13}$$

which transforms (18.2.9) into

$$\begin{pmatrix}\dot u\\ \dot v\\ \dot w\end{pmatrix}=\begin{pmatrix}0&0&0\\0&-(1+\sigma)&0\\0&0&-\beta\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}+\frac{1}{1+\sigma}\begin{pmatrix}\sigma\bar\rho(u+\sigma v)-\sigma w(u+\sigma v)\\-\bar\rho(u+\sigma v)+w(u+\sigma v)\\(1+\sigma)(u+\sigma v)(u-v)\end{pmatrix},$$
$$\dot{\bar\rho}=0. \tag{18.2.14}$$

Thus, from center manifold theory, the stability of $(x,y,z)=(0,0,0)$ near $\bar\rho=0$ can be determined by studying a one-parameter family of first-order ordinary differential equations on a center manifold, which can be represented as a graph over the u and $\bar\rho$ variables, i.e.,

$$W^c(0)=\Big\{(u,v,w,\bar\rho)\in\mathbb{R}^4\mid v=h_1(u,\bar\rho),w=h_2(u,\bar\rho),$$
$$h_i(0,0)=0,Dh_i(0,0)=0,i=1,2\Big\} \tag{18.2.15}$$

for u and $\bar\rho$ sufficiently small.

We now want to compute the center manifold and derive the vector field on the center manifold. Using Theorem 18.1.4, we assume

$$h_1(u,\bar\rho)=a_1u^2+a_2u\bar\rho+a_3\bar\rho^2+\cdots,$$
$$h_2(u,\bar\rho)=b_1u^2+b_2u\bar\rho+b_3\bar\rho^2+\cdots. \tag{18.2.16}$$

Recall from (2.1.27) that the center manifold must satisfy

$$\mathcal{N}\Big(h(x,\varepsilon)\Big)=D_xh(x,\varepsilon)\left[Ax+f(x,h(x,\varepsilon),\varepsilon)\right]$$
$$-Bh(x,\varepsilon)-g(x,h(x,\varepsilon),\varepsilon)=0, \tag{18.2.17}$$

where, in this example,

$$x\equiv u,\qquad y\equiv(v,w),\qquad \varepsilon\equiv\bar\rho,\qquad h=(h_1,h_2),$$
$$A=0,$$
$$B=\begin{pmatrix}-(1+\sigma)&0\\0&-\beta\end{pmatrix}, \tag{18.2.18}$$
$$f(x,y,\varepsilon)=\frac{1}{1+\sigma}[\sigma\bar\rho(u+\sigma v)-\sigma w(u+\sigma v)],$$
$$g(x,y,\varepsilon)=\frac{1}{1+\sigma}\begin{pmatrix}-\bar\rho(u+\sigma v)+w(u+\sigma v)\\(1+\sigma)(u+\sigma v)(u-v)\end{pmatrix}.$$

Substituting (18.2.16) into (18.2.17) and using (18.2.19) gives the two components of the equation for the center manifold.

$$(2a_1u+a_2\bar\rho+\cdots)\left[\frac{\sigma}{1+\sigma}\Big(\bar\rho(u+\sigma h_1)-h_2(u+\sigma h_1)\Big)\right]$$
$$+(1+\sigma)h_1+\frac{\bar\rho}{1+\sigma}(u+\sigma h_1)-\frac{h_2}{1+\sigma}(u+\sigma h_1)=0,$$

$$(2b_1u+b_2\bar\rho+\cdots)\left[\frac{\sigma}{1+\sigma}\Big(\bar\rho(u+\sigma h_1)-h_2(u+\sigma h_1)\Big)\right]$$
$$+\beta h_2-(u+\sigma h_1)(u-h_1)=0. \tag{18.2.19}$$

Equating terms of like powers to zero gives

$$u^2:a_1(1+\sigma)=0\Rightarrow a_1=0,$$
$$\beta b_1-1=0\Rightarrow b_1=\frac{1}{\beta}, \tag{18.2.20}$$
$$u\bar\rho:(1+\sigma)a_2+\frac{1}{1+\sigma}=0\Rightarrow a_2=\frac{-1}{(1+\sigma)^2},$$
$$\beta b_2=0\Rightarrow b_2=0.$$

Then, using (18.2.21) and (18.2.16), we obtain

$$h_1(u,\bar\rho)=-\frac{1}{(1+\sigma)^2}u\bar\rho+\cdots,$$
$$h_2(u,\bar\rho)=\frac{1}{\beta}u^2+\cdots. \tag{18.2.21}$$

Finally, substituting (18.2.21) into (18.2.14) we obtain the vector field reduced to the center manifold

$$\dot u=\frac{\sigma}{1+\sigma}u\left(\bar\rho-\frac{1}{\beta}u^2+\cdots\right),$$
$$\dot{\bar\rho}=0. \tag{18.2.22}$$

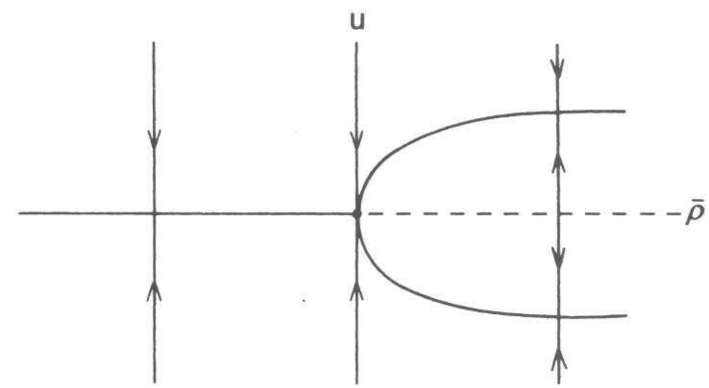

FIGURE 18.2.1.

In Figure 18.2.1 we plot the fixed points of (18.2.22) *neglecting* higher order terms such as $\mathcal{O}(\bar\rho^2)$, $\mathcal{O}(u\bar\rho^2)$, $\mathcal{O}(u^3)$, etc. It should be clear that $u=0$ is always a fixed point and is stable for $\bar\rho<0$ and unstable for $\bar\rho>0$. At the point of exchange of stability (i.e., $\bar\rho=0$) two new stable fixed points are created and are given by

$$\bar\rho=\frac{1}{\beta}u^2. \tag{18.2.23}$$

A simple calculation shows that these fixed points are stable. In Chapter 20 we will see that this is an example of a *pitchfork bifurcation*.

Before leaving this example two comments are in order.

1. Figure 18.2.1 shows the advantage of introducing the parameter as a new dependent variable. In a full neighborhood in parameter space new solutions are "captured" on the center manifold. In Figure 18.2.1, for each fixed $\bar{\rho}$ we have a flow in the u direction; this is represented by the vertical lines with arrows.
2. We have not considered the effects of the higher order terms in (18.2.22) on Figure 18.2.1. In Chapter 20 we will show that they do not qualitatively change the figure (i.e., they do not create, destroy, or change the stability of any of the fixed points) near the origin.

End of Example 18.2.1

18.3 The Inclusion of Linearly Unstable Directions

Suppose we consider the system

$$\begin{aligned} \dot{x} &= Ax + f(x, y, z), \\ \dot{y} &= By + g(x, y, z), \\ \dot{z} &= Cz + h(x, y, z), \qquad (x, y, z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u, \end{aligned} \tag{18.3.1}$$

where

$$\begin{aligned} f(0,0,0) &= 0, \qquad Df(0,0,0) = 0, \\ g(0,0,0) &= 0, \qquad Dg(0,0,0) = 0, \\ h(0,0,0) &= 0, \qquad Dh(0,0,0) = 0, \end{aligned}$$

and f, g, and h are $\mathbf{C}^r (r \geq 2)$ in some neighborhood of the origin, A is a $c \times c$ matrix having eigenvalues with zero real parts, B is an $s \times s$ matrix having eigenvalues with negative real parts, and C is a $u \times u$ matrix having eigenvalues with positive real parts.

In this case $(x, y, z) = (0, 0, 0)$ is unstable due to the existence of a u-dimensional unstable manifold. However, much of the center manifold theory still applies, in particular Theorem 18.1.2 concerning existence, with the center manifold being locally represented by

$$\begin{aligned} W^c(0) = \big\{(x, y, z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u \mid y = h_1(x), z = h_2(x), \\ h_i(0) = 0, Dh_i(0) = 0, i = 1, 2\big\} \end{aligned} \tag{18.3.2}$$

for x sufficiently small. The vector field restricted to the center manifold is given by

$$\dot{u} = Au + f\big(u, h_1(u), h_2(u)\big), \qquad u \in \mathbb{R}^c. \tag{18.3.3}$$

Using the fact that the center manifold is invariant under the dynamics generated by (18.3.1), we obtain

$$\begin{aligned} \dot{x} &= Ax + f\big(x, h_1(x), h_2(x)\big), \\ \dot{y} &= Dh_1(x)\dot{x} = Bh_1(x) + g\big(x, h_1(x), h_2(x)\big), \\ \dot{z} &= Dh_2(x)\dot{x} = Ch_2(x) + h\big(x, h_1(x), h_2(x)\big), \end{aligned} \tag{18.3.4}$$

which yields the following quasilinear partial differential equation for $h_1(x)$ and $h_2(x)$

$$\begin{aligned} & Dh_1(x)\big[Ax + f\big(x, h_1(x), h_2(x)\big)\big] \\ & \qquad - Bh_1(x) - g\big(x, h_1(x), h_2(x)\big) = 0, \\ & Dh_2(x)\big[Ax + f\big(x, h_1(x), h_2(x)\big)\big] \\ & \qquad - Ch_2(x) - h\big(x, h_1(x), h_2(x)\big) = 0. \end{aligned} \tag{18.3.5}$$

Theorem 18.1.4 also holds in order that we may justify solving (18.3.5) approximately via power series expansions. We can also include parameters in exactly the same way as in Section 18.2.

Of course, Theorem 18.1.3 does not hold as a result of the presence of the exponentially linearly unstable directions. Nevertheless, the formulation of the theory with the inclusion of the linearly unstable directions is still useful. It is often important to know the nature of solutions having saddle-type stability since their stable manifolds may play a role in forming the boundaries of the basins of attraction of attracting sets. In the context of bifurcation theory, the creation of unstable solutions may be important since it may be possible for them to undergo secondary bifurcations and, consequently, become stable.

18.4 Center Manifolds for Maps

The center manifold theory can be modified so that it applies to maps with only a slight difference in the method by which the center manifold is calculated. We outline the theory below.

Suppose we have the map

$$\begin{aligned} x &\longmapsto Ax + f(x, y), \\ y &\longmapsto By + g(x, y), \qquad (x, y) \in \mathbb{R}^c \times \mathbb{R}^s, \end{aligned} \tag{18.4.1}$$

or

$$\begin{aligned} x_{n+1} &= Ax_n + f(x_n, y_n), \\ y_{n+1} &= By_n + g(x_n, y_n), \end{aligned}$$

where

$$f(0,0)=0, \qquad Df(0,0)=0,$$
$$g(0,0)=0, \qquad Dg(0,0)=0,$$

and f and g are $\mathbf{C}^r (r \geq 2)$ in some neighborhood of the orgin, A is a $c \times c$ matrix with eigenvalues of modulus one, and B is an $s \times s$ matrix with eigenvalues of modulus less than one.

Evidently $(x,y)=(0,0)$ is a fixed point of (18.4.1), and the linear approximation is not sufficient for determining its stability. We have the following theorems, which are completely analogous to Theorems 18.1.2, 18.1.3, and 18.1.4.

Theorem 18.4.1 (Existence) *There exists a* $\mathbf{C}^r$ *center manifold for (18.4.1) which can be locally represented as a graph as follows*

$$W^c(0) = \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x), |x| < \delta, h(0)=0, Dh(0)=0\} \tag{18.4.2}$$

for δ *sufficiently small. Moreover, the dynamics of (18.4.1) restricted to the center manifold is, for* u *sufficiently small, given by the c-dimensional map*

$$u \longmapsto Au + f\big(u, h(u)\big), \qquad u \in \mathbb{R}^c. \tag{18.4.3}$$

Proof: See Carr [1981]. □

The next theorem allows us to conclude that $(x,y)=(0,0)$ is stable or unstable based on whether or not $u=0$ is stable or unstable in (18.4.3).

Theorem 18.4.2 (Stability) i) *Suppose the zero solution of (18.4.3) is stable (asymptotically stable) (unstable). Then the zero solution of (18.4.1) is stable (asymptotically stable) (unstable).* ii) *Suppose that the zero solution of (18.4.3) is stable. Let* (x_n, y_n) *be a solution of (18.4.1) with* (x_0, y_0) *sufficiently small. Then there is a solution* u_n *of (18.4.3) such that* $|x_n - u_n| \leq k\beta^n$ *and* $|y_n - h(u_n)| \leq k\beta^n$ *for all* n *where* k *and* β *are positive constants with* $\beta < 1$.

Proof: See Carr [1981]. □

Next we want to compute the center manifold so that we can derive (18.4.3). This is done in exactly the same way as for vector fields, i.e., by deriving a nonlinear functional equation that the graph of $h(x)$ must satisfy in order for it to be invariant under the dynamics generated by (18.4.1). In this case we have

$$\begin{aligned} x_{n+1} &= Ax_n + f\big(x_n, h(x_n)\big), \\ y_{n+1} &= h(x_{n+1}) = Bh(x_n) + g\big(x_n, h(x_n)\big), \end{aligned} \tag{18.4.4}$$

or

$$\mathcal{N}\big(h(x)\big) = h\big(Ax + f\big(x, h(x)\big)\big) - Bh(x) - g\big(x, h(x)\big) = 0. \tag{18.4.5}$$

(Note: the reader should compare (18.4.5) with (18.1.10).) The next theorem justifies the approximate solution of (18.4.5) via power series expansions.

Theorem 18.4.3 (Approximation) *Let* $\phi : \mathbb{R}^c \to \mathbb{R}^s$ *be a* $\mathbf{C}^1$ *map with* $\phi(0)=0, \phi'(0)=0$*, and* $\mathcal{N}\big(\phi(x)\big) = \mathcal{O}(|x|^q)$ *as* $x \to 0$ *for some* $q > 1$*. Then*

$$h(x) = \phi(x) + \mathcal{O}(|x|^q) \qquad \text{as } x \to 0.$$

Proof: See Carr [1981]. □

We now give an example.

Example 18.4.1. Consider the map

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} vw \\ u^2 \\ -uv \end{pmatrix}, \quad (u,v,w) \in \mathbb{R}^3. \tag{18.4.6}$$

It should be clear that $(u,v,w)=(0,0,0)$ is a fixed point of (18.4.6), and the eigenvalues associated with the map linearized about this fixed point are $-1, -\frac{1}{2}, \frac{1}{2}$. Thus, the linear approximation does not suffice to determine the stability or instability. We will apply center manifold theory to this problem.

The center manifold can locally be represented as follows

$$W^c(0) = \Big\{(u,v,w) \in \mathbb{R}^3 \mid v = h_1(u), w = h_2(u), h_i(0)=0, Dh_i(0)=0, i=1,2\Big\} \tag{18.4.7}$$

for u sufficiently small. Recall that the center manifold must satisfy the following equation

$$\mathcal{N}\Big(h(x)\Big) = h\Big(Ax + f\Big(x, h(x)\Big)\Big) - Bh(x) - g\Big(x, h(x)\Big) = 0, \tag{18.4.8}$$

where, in this example,

$$\begin{aligned} x = u, \quad y &\equiv (v,w), \quad h = (h_1, h_2), \\ A &= -1, \\ B &= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ f(u,v,w) &= vw, \\ g(u,v,w) &= \begin{pmatrix} u^2 \\ -uv \end{pmatrix}. \end{aligned} \tag{18.4.9}$$

We assume a center manifold of the form

$$h(u) = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} = \begin{pmatrix} a_1u^2 + b_1u^3 + \mathcal{O}(u^4) \\ a_2u^2 + b_2u^3 + \mathcal{O}(u^4) \end{pmatrix}. \tag{18.4.10}$$