06170420010.

Example Consider the system

$$\dot{x} = -2\cos x - \cos y$$
$$\dot{y} = -2\cos y - \cos x.$$

Show it is reversible, but not conservative. Plot the phase portrait.

Soln: * Consider $t \longrightarrow -t$
the change of variable $R\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \end{pmatrix}$.

The system is invariant, then is reversible.

* Remember that a conservative system does not have attractive fixed points.

* Let us find the fixed points: (x_f, y_f): $\dot{x}_f = 0$, $\dot{y}_f = 0$.

$$-2\cos x_f - \cos y_f = 0 \quad \ldots\ldots (1)$$
$$-2\cos y_f - \cos x_f = 0 \quad \ldots\ldots (2)$$

(1) implies: (2)

$$2\cos x_f = -\cos y_f = \frac{1}{2}\cos x_f \Rightarrow 4\cos x_f = \cos x_f$$
$$\Rightarrow \boxed{\cos x_f = 0}$$
$$\overset{(1)\text{ or }(2)}{\Longrightarrow} \boxed{\cos y_f = 0}$$

Then, the fixed pts are: $\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} \pm \pi/2 \\ \pm \pi/2 \end{pmatrix}$, plus a 2π periodicity in the x- and y-directions.

Compute now the Jacobian matrix:

$$J(x,y) = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}$$

* Evaluate at $(x_f, y_f) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: $J = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

Here; $\tau = -4$, $\Delta = 3$, $\tau^2 - 4\Delta = 4 > 0$.

This means $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a stable node; therefore, the system cannot be conservative.

* Evaluate at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$: $J = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\tau = 4$, $\Delta = 3$
$\tau^2 - 4\Delta = 4 > 0$

This is an <u>unstable</u> node.

* Evaluate at $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: $J = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$: $\tau = 0$
$\Delta = -4 + 1 = -3$
$= -3 < 0$

$\Delta < 0 \Rightarrow$ This is a <u>saddle</u>:

* Evaluate. $\left(+\frac{\pi}{2}, -\frac{\pi}{2}\right)$: $J = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$: $\tau = 0$
$\Delta = -4 + 1 = -3$

$\Delta < 0$, it is a <u>saddle</u>:

---

* For: $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$: $J = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$.

<u>Attracting node</u>:

<u>Eigenvalues</u>: $\det\begin{pmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{pmatrix} = 0$ $\quad (\lambda+2)^2 - 1 = 0$

$$\Rightarrow \lambda = -2 \pm 1 = \begin{cases} -1, & \leftarrow \text{slow decay} \\ -3, & \leftarrow \text{fast decay} \end{cases}$$

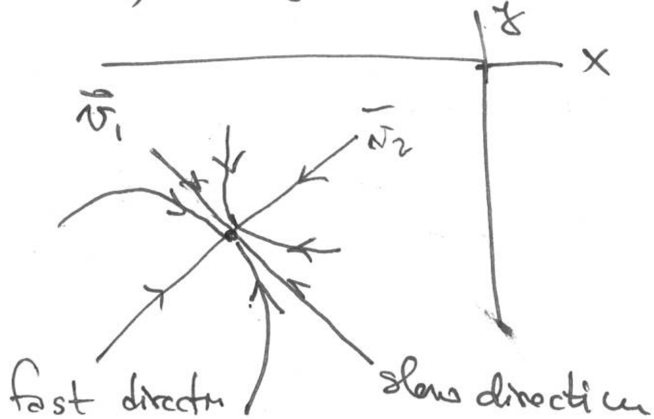
Eigenvectors:

$\lambda_1 = -1$ $\quad (J - \lambda_1 I)\vec{v}_1 = \begin{pmatrix} -2+1 & -1 \\ -1 & -2+1 \end{pmatrix}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -v_1 - w_1 = 0$$

Choose: $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Remember that, when $t \to \infty$, trajectories are tangent to the slow direction:



$\lambda_2 = -2$: $\quad (J - \lambda_2 I)\bar{v}_2 = \begin{pmatrix} -2+3 & -1 \\ -1 & -2+3 \end{pmatrix}\bar{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 - w_2 = 0 \Rightarrow \boxed{\bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

---

* For $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we know we have a saddle.

Eigenvalues: $J = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$

$$\det(J - \lambda I) = \det\begin{pmatrix} -2-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} = 0 \Rightarrow (\lambda+2)(\lambda-2) + 1 = 0$$

$$\Rightarrow \lambda^2 - 4 + 1 = 0$$

$$\Rightarrow \lambda = \pm\sqrt{3}$$

Unstable manifold corresponds to: $\lambda_2 = +\sqrt{3}$:

$$(J - \lambda_2 I)\vec{v}_2 = \begin{pmatrix} -2-\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

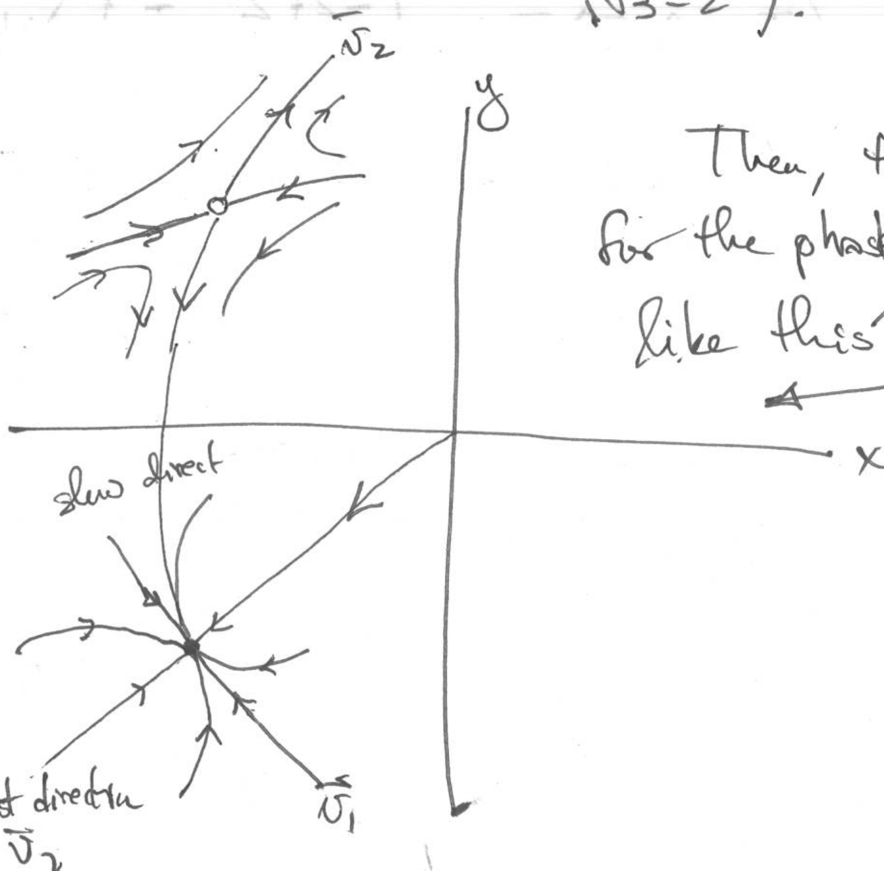
$$\Rightarrow \quad -(2+\sqrt{3})\, v_2 + w_3 = 0$$

$$\Rightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix}$$

Stable manifold $\longleftrightarrow$ $\lambda_1 = -\sqrt{3}$ ::

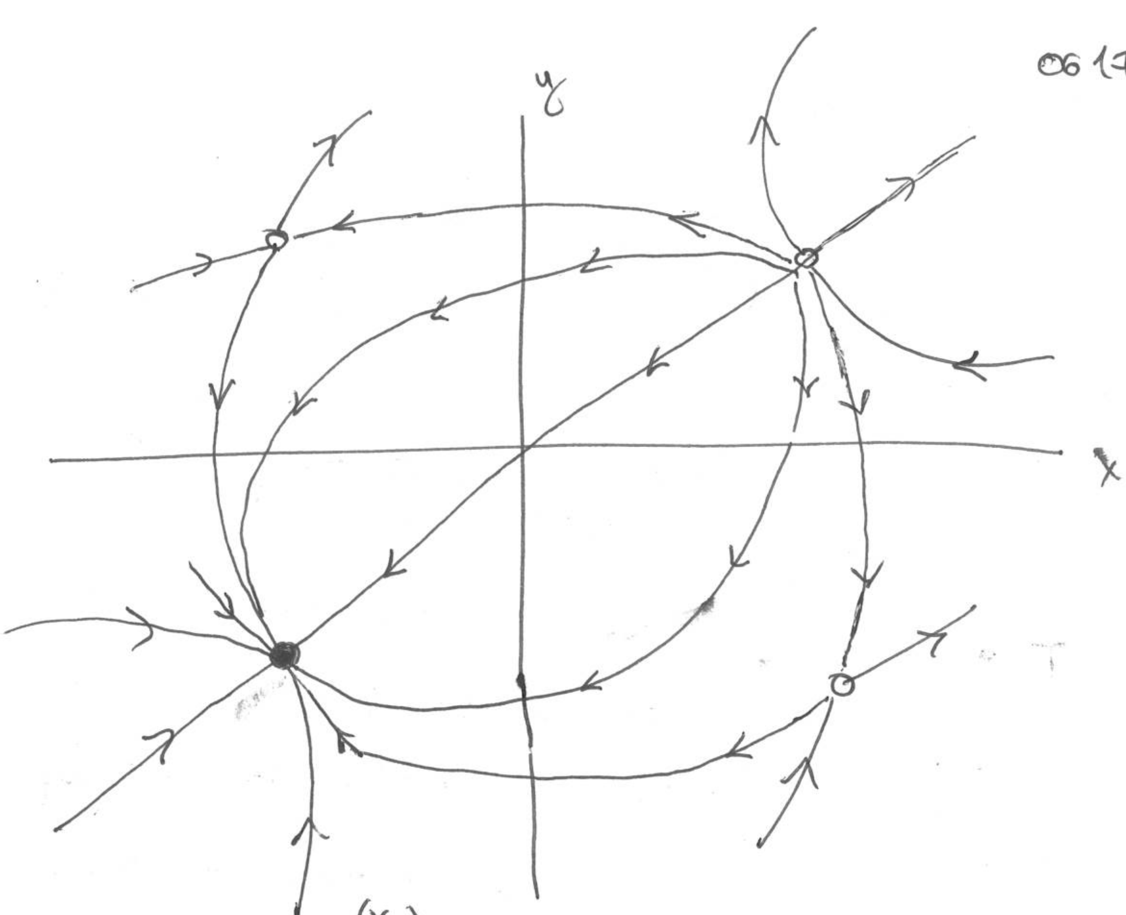
$$(J - \lambda_1 I)\vec{v}_1 = \begin{pmatrix} -2+\sqrt{3} & 1 \\ -1 & 2+\sqrt{3} \end{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(\sqrt{3}-2)v_1 + w_1 = 0 \quad \Rightarrow \quad \vec{v}_1 = \begin{pmatrix} -1 \\ \sqrt{3}-2 \end{pmatrix}.$$



Then, the first sketch for the phase portrait looks like this

06 17042010.



Using the symmetry $\overset{(x,y)}{\longrightarrow} R(x,y) = (-x,-y)$, we can plot the trajectories for $x \geq 0$.

We reverse the direction of the arrows due to the symmetry:

$$t \longrightarrow -t.$$

This system models two superconductors in Josephson junctions coupled through a resistive load (Tsang, et. al. 1991). See

See ~~exercise~~ 6.6.9.

Example 8.7.4.

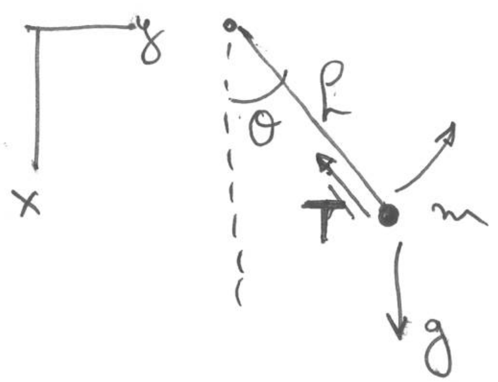
Reversible, nonconservative systems arise in

lasers: (Politi, et. al 1986)

fluid flows (Stone Nadim Strogatz 1991
Exercise 6.6.8)

6.7 The Pendulum.

The equation of a pendulum with length "L", and subject to the force of gravity "g", is given by:



$$L \frac{d^2\theta}{dt^2} = -g \sin\theta, \qquad (1)$$

where θ is the angle with respect to the vertical; m – mass of the pendulum. Notice the equation does not depend on m

To derive the equation of the pendulum, we invoke Newton's law:

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + = \begin{pmatrix} mg \\ 0 \end{pmatrix} + \begin{pmatrix} -T\cos\theta \\ -T\sin\theta \end{pmatrix},$$

where T is the tension suffered by the particle due to the stick of the pendulum.

If we now write (x, y) in terms of θ:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = L \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}$$

follows that:

$$\begin{pmatrix} -mL\cos\theta\, \theta'^2 + (-mL)\sin\theta\, \theta'' \\ -mL\sin\theta\, \theta'^2 + mL\cos\theta\, \theta'' \end{pmatrix} = \begin{pmatrix} mg \\ 0 \end{pmatrix} + (-T)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

where $\theta' = \frac{d}{dt}\theta$

or:

$$-mL\theta'^2\,\vec{v}_1 + mL\theta''\,\vec{v}_2 = \begin{pmatrix} mg \\ 0 \end{pmatrix} - T\,\vec{v}_1 \qquad (*)$$

where $\vec{v}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$; and $\vec{v}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

Notice that: $(mg, 0)^t$ can be written as:

$$\begin{pmatrix} mg \\ 0 \end{pmatrix} = mg\cos\theta\,\vec{v}_1 - mg\sin\theta\,\vec{v}_2 .$$

Therefore, eq. $(*)$ above becomes:

$$\left(-mL\theta'^2 - mg\cos\theta + T\right)\vec{v}_1 + \left(mL\theta'' + mg\sin\theta\right)\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, we obtain the system of equations:

$$\boxed{-mL\theta'^2 - mg\cos\theta + T = 0} \qquad \ldots (2)$$

$$\boxed{mL\theta'' + mg\sin\theta = 0} \qquad \ldots (3)$$

Notice that equation (2) provides the value for T, once $\theta = \theta(t)$ is known.

Equation (3) provides a description for the motion of the angle θ of the pendulum. This is the <u>equation of the pendulum</u>.

Notice that eq. (3) is independent of the mass m, ($m \neq 0$) once we divide by m, turning to be eq. (1) in page 182.

Then, equation (1) is the equation of the pendulum

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta}$$

For $\theta \sim 0$, we get an approximated equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta,$$

with solution: $\theta(t) = \theta_0 \cos(\omega t) + \frac{V_0}{\omega}\sin(\omega t)$,

with $\theta_0 \equiv \theta(0)$ and $V_0 \equiv \theta'(0)$, and

$\boxed{\omega \equiv \sqrt{\frac{g}{L}}}$ is the natural frequency of the pendulum.

We can adimensionalize the equation of the pendulum by setting and defining the new variable:

$\tau = \omega t$, which is dimensionless.

The equation becomes: $\boxed{\frac{d^2\theta}{d\tau^2} = -\sin\theta}$ $\ddot{\theta} = -\sin\theta$

Written as a system:

$$\boxed{\begin{aligned} \dot{\theta} &= V \\ \dot{V} &= -\sin\theta \end{aligned}}$$

, where $\frac{d}{d\tau}(\ ) = (\ )^{\bullet}$

06 17 04 2010

Fixed points. $\dot{\theta}_f = 0 \Rightarrow v_f = 0$
$\dot{v}_f = 0 \quad -\sin\theta_f = 0.$

Follows that: $(\theta_f, v_f) = (0, 0)$ and $(\pi, 0)$, plus 2π-periodic θ fixed pts.

The Jacobian matrix is:

$$J(\theta, v) = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}.$$

At the origin: $$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Where $\tau = 0$, $\Delta = 1$, and $\tau^2 - 4\Delta = -4 < 0$.

The pendulum has a ~~linear~~ center, according to linear analysis.

This center is actually a center for the nonlinear system as well.

(1) The system is reversible: under the following transformation $\tau \longrightarrow -\tau$, $v \longrightarrow -v$.

(2) The system is conservative.

From the equation
$$\frac{d^2}{d\tau^2}\theta + \sin\theta = 0$$
follows
$$\frac{d\theta}{d\tau}\frac{d^2}{d\tau^2}\theta + \sin\theta\frac{d\theta}{d\tau} = 0.$$

By the chain rule:

$$\frac{d}{d\tau}\left(\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 - \cos\theta\right) = 0$$

Hence:

$$\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 - \cos\theta = \text{const.}$$

This is the conservation of energy:

$$T = \frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2$$

$$V(\theta) \equiv 1 - \cos\theta.$$

are the non-dimensional kinetic and potential energies.
We have choosen $V(\theta)$ to be non-negative. Then:

$$\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 + 1 - \cos\theta = \underbrace{1 + \text{const}}_{E}.$$

Hence

$$\boxed{\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 + V(\theta) = E}$$

Notice that the energy can be written as a function of the position and generalized momentum (the generalized coordinates).

$$E(\theta, v) = \frac{1}{2}v^2 + V(\theta) = E.$$

Notice that $\nabla E = \begin{pmatrix} V'(\theta) \\ v \end{pmatrix} = \begin{pmatrix} \sin\theta \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ at $(\theta, v) = (0,0)$.

Similarly $\frac{\partial^2 E}{\partial u \partial v} = \begin{pmatrix} E_{\theta\theta} & E_{\theta v} \\ E_{v\theta} & E_{vv} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ 0 & 1 \end{pmatrix}$

At $(\theta, \nu) = (0,0)$

$$\frac{\partial^2 E}{\partial u \partial v} J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and } \det J = 1 > 0.$$

Then, we have a minimum (at the origin) for the energy. Then, the origin is a <u>center</u> for the nonlinear system. (This is the second proof that the system possesses a nonlinear center).

Similarly, for $(\theta, \nu) \sim (0,0)$

$$E \cong \frac{1}{2}\left(\nu^2 + \theta^2\right),$$ ← this is direct from the nonlinear

i.e. near the origin, trajectories are approximated by circles

— c —

Fixed point at $(\pi, 0)$. The Jacobian at $(\theta, \nu) = (\pi, 0)$ is:

$$J(\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}\Bigg|_{(\pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\Delta = -1$. Then, it is a <u>saddle.</u>

The eigenvalues are $\lambda = \pm 1$.

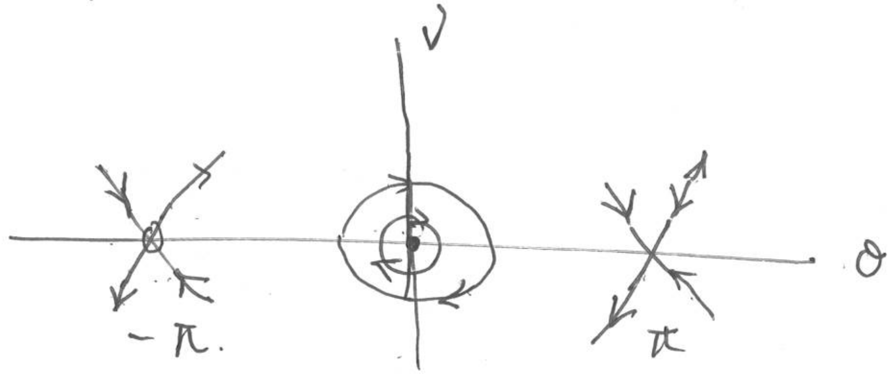
For $\lambda_1 = -1$, the eigenvector $\vec{v}_1$ is:

$$\begin{pmatrix} +1 & 1 \\ 1 & +1 \end{pmatrix} \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}!$$ stable manifold.

For $\lambda_2 = 1$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Unstable manifold.

The phase portrait can be plotted, so far, near the fixed points:



and it's periodically repeated every 2π in the θ-direction.

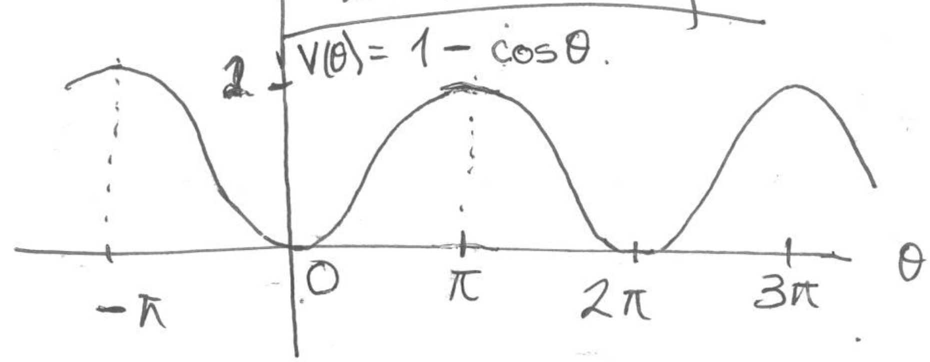
Notice that if $V > 0$, then $\dot{\theta} \geq 0$, and the flux is to the rigth for $V>0$, i.e., in the upper-half plane.

Now, from the law of conservation's law of energy:

$$E = \frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 + V(\theta) \geq V(\theta)$$

follows that, ~~under trajectories~~, the only trajectories allowed are those such that the following inequality holds:

$$\boxed{E \geq V(\theta)}$$



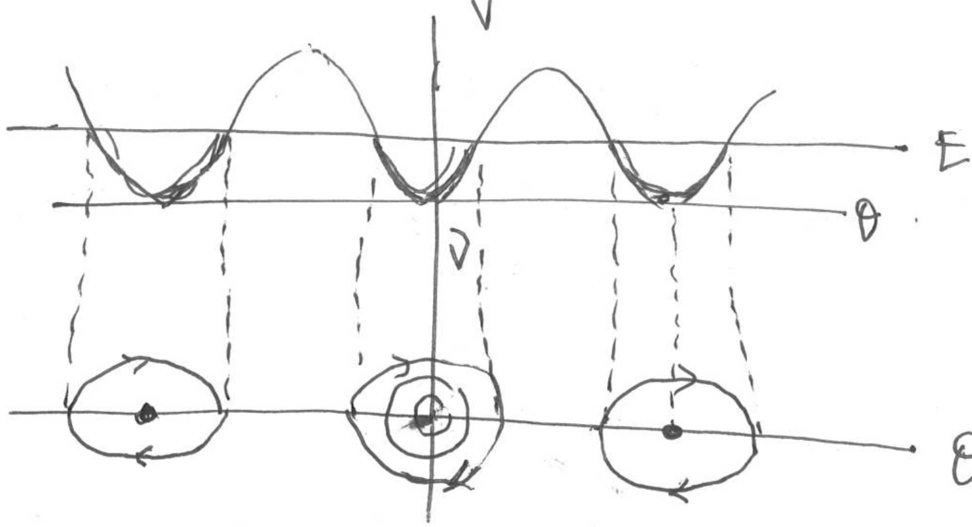
Trajectories for which $E<0$ are not allowed, since that implies $E < V(\theta)$ (but we require $V(\theta) \leq E$).

If $E=0$, then:

$$E=0 \geq V(\theta) \geq 0.$$

Then $V(\theta)=0 \Rightarrow \theta = 0, \pm 2\pi, \dots$

If $0 < E < 2$:

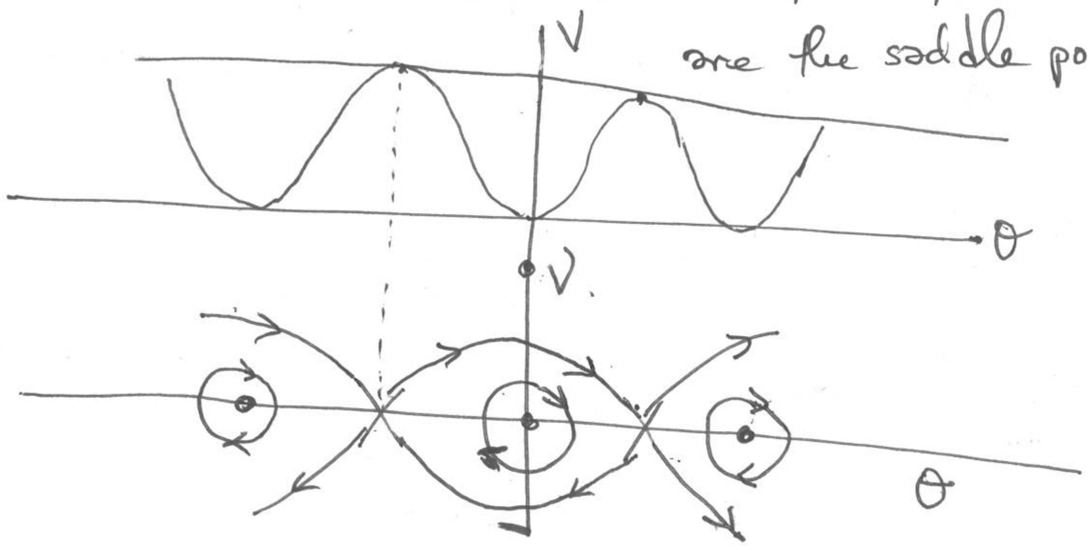


We observe that the trajectories are bounded.

If $E = 2$, $\Rightarrow E = V(\theta) = 1-\cos\theta$, if $\theta_c = 0$.

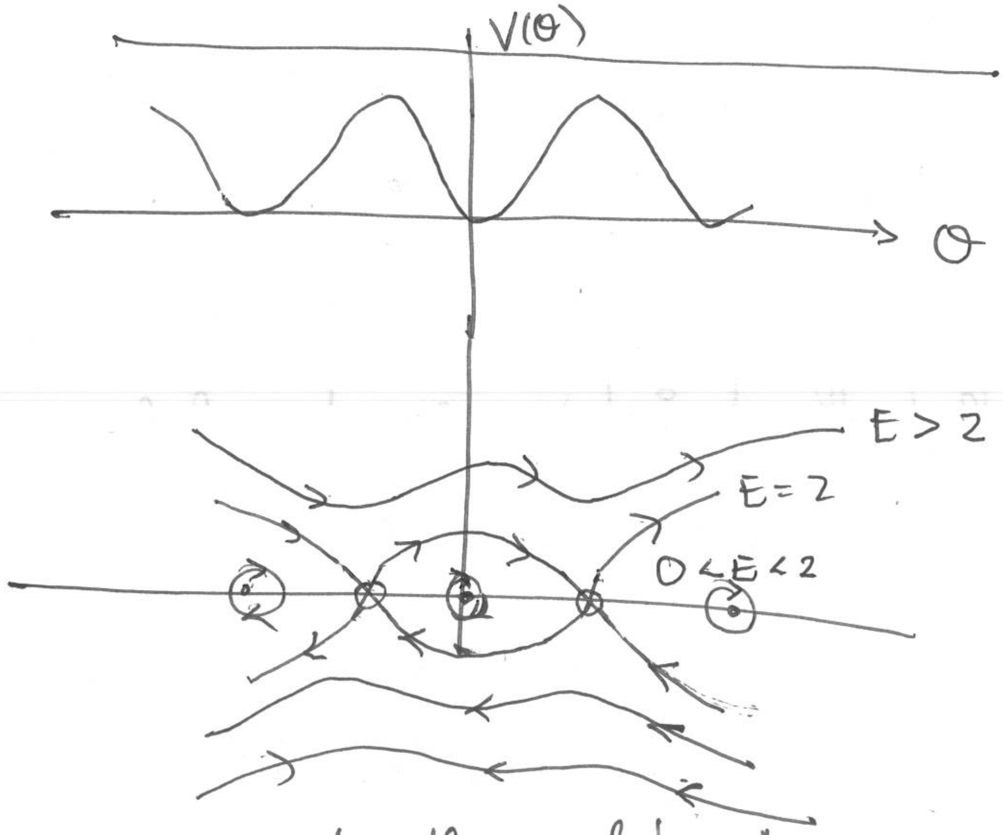
Then: $2 = 1 - \cos\theta \Rightarrow \cos\theta = -1$

$\Rightarrow \theta = -\pi, -3\pi, \dots$ (mod 2π), are the saddle points.



If $E = 2$ and $\theta_c \neq 0$, then the phase portrait present curves that join two saddle points.
They are called heteroclinic orbits, since they join two different fixed points.

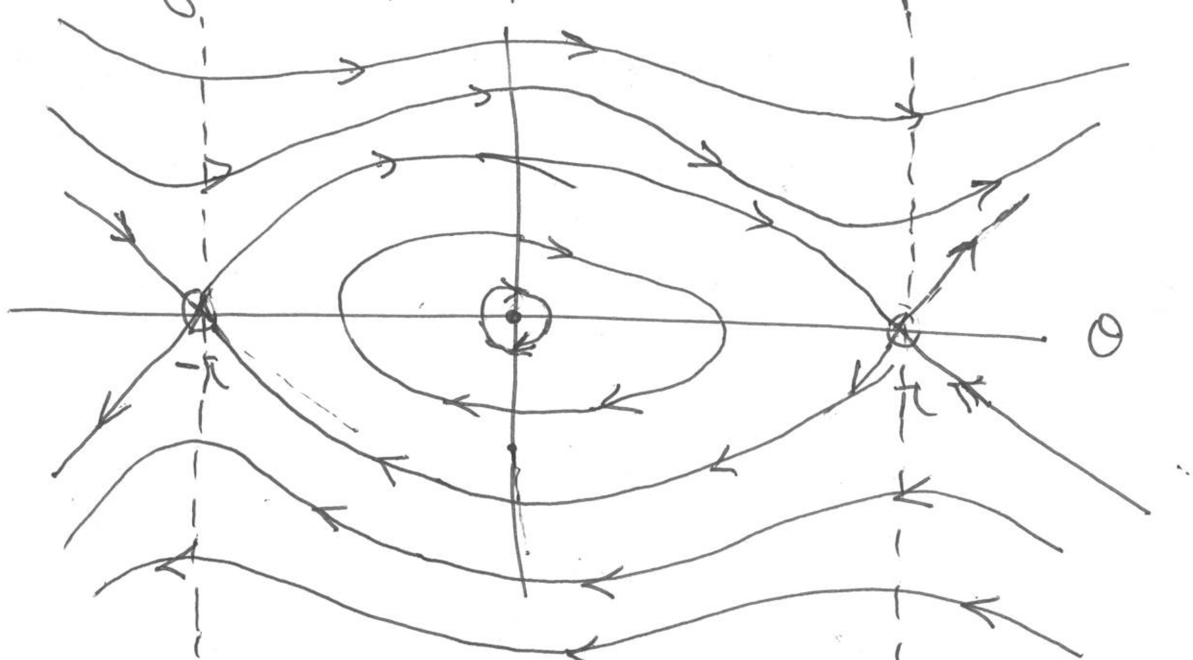
Finally, for $E > 2$, the trajectories are unbounded, since $E > 2 \geq 1 - \cos\theta = V(\theta)$, $\forall \theta$.



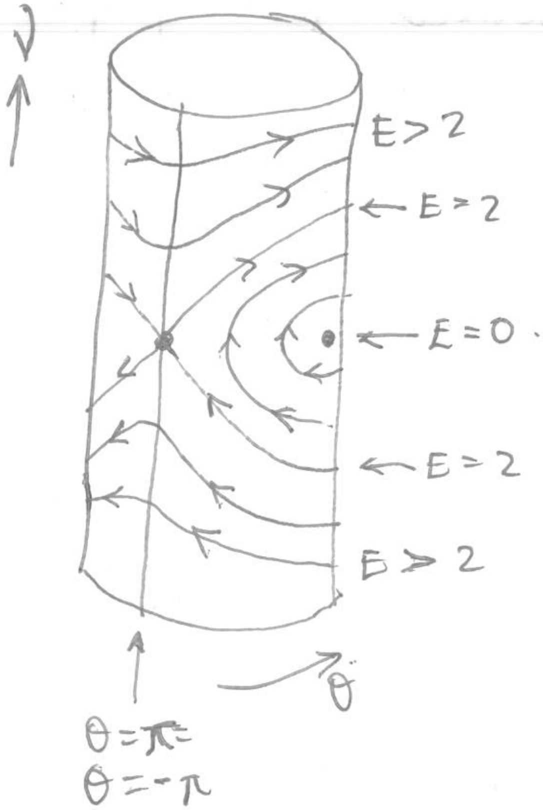
This corresponds to the pendulum turning around the pivot all the time.

Notice that the heteroclinic orbit separates the periodic, bounded behaviour, to that one that goes around all the time. For this reason, the heteroclinic orbit is also called the separatrix trajectory.

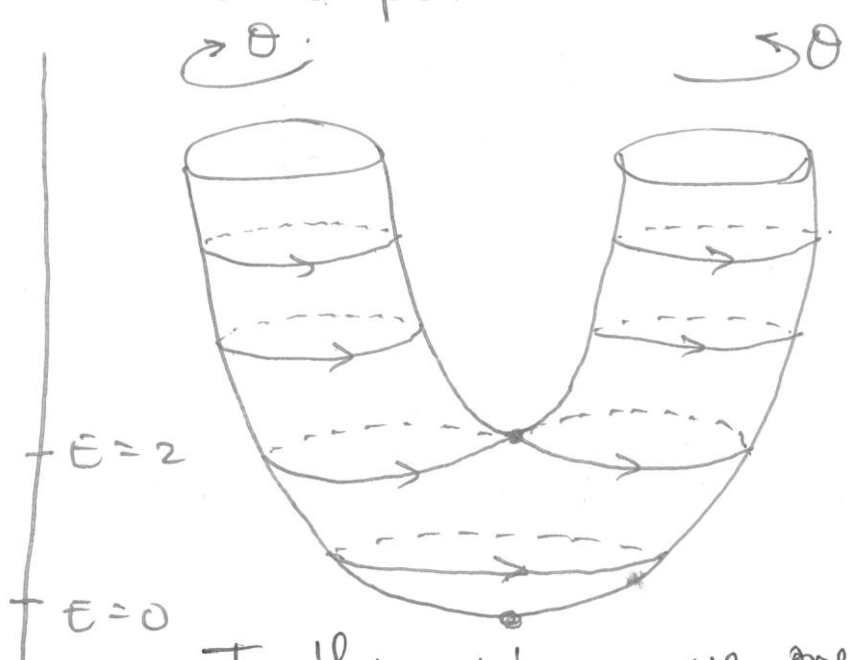
Finally, the phase portrait for the pendulum looks like:



Now, since the pendulum is 2π-periodic, we can cut the phase portrait at $\theta = \pi$ and $\theta = -\pi$, and glue it together to get the cylindrical phase space:



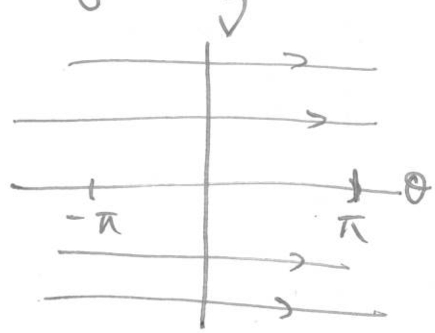
We can bent the cylinder in-to a U-shape:

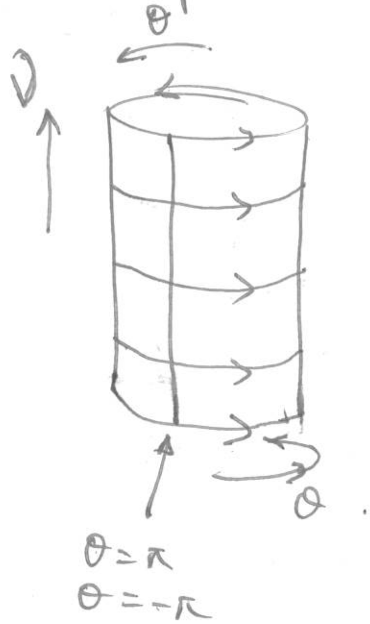


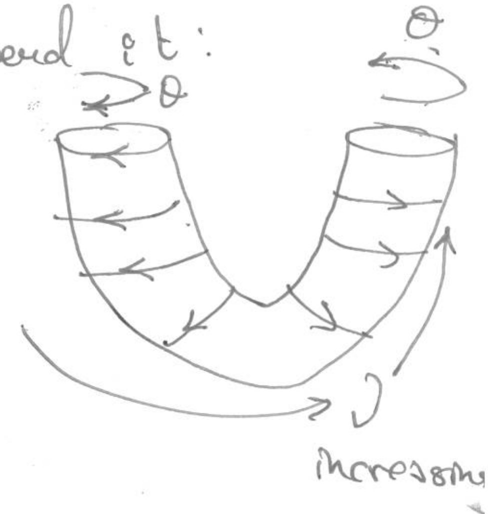
In this pictures, we are plotting the trajectories (not the coordinate system).

Notice that in this previous figure, we are plotting the trajectories, but not the coordinate system..

If we would like to plot the coordinate system,

we go from: 

to: 

and bend it: 

Damping. Friction forces are assumed to be proportional to the velocity of the particle in motion, and opposed to the direction of movement:

$$F_{frict} \sim -\hat{b} v ,$$

Now: $v = L\dot{\theta}$ hence: $F_{frict} = -\hat{b} L \dot{\theta} = -b\dot{\theta}$,

all of this in dimensionless variables.

Then, the equation of motion for the damped pendulum is:

$$\ddot{\theta} = -\sin\theta - b\dot{\theta} .$$

06/04/2010

Notice that:

$$\ddot{\theta} + \sin\theta = -b\dot{\theta}$$

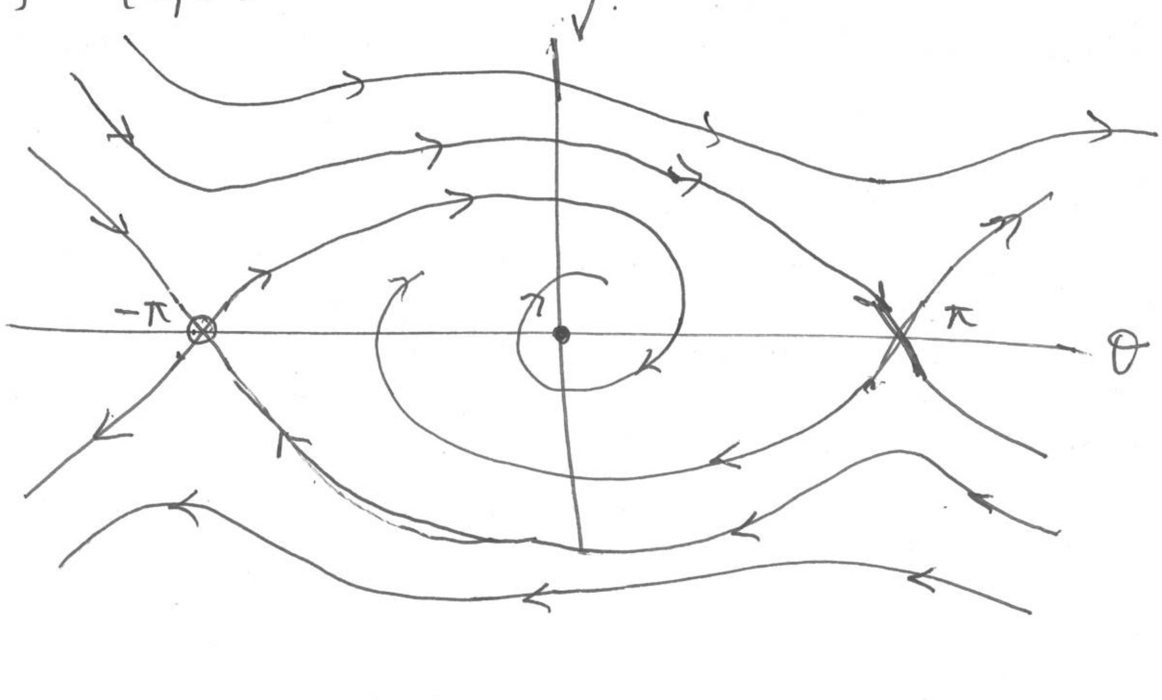
$$\Rightarrow \quad \dot{\theta}\ddot{\theta} + \sin\theta\,\dot{\theta} = -b\dot{\theta}^2$$

$$\Rightarrow \quad \frac{d}{d\tau}\left(\frac{\dot{\theta}^2}{2} + (1-\cos\theta)\right) = -b\dot{\theta}^2$$

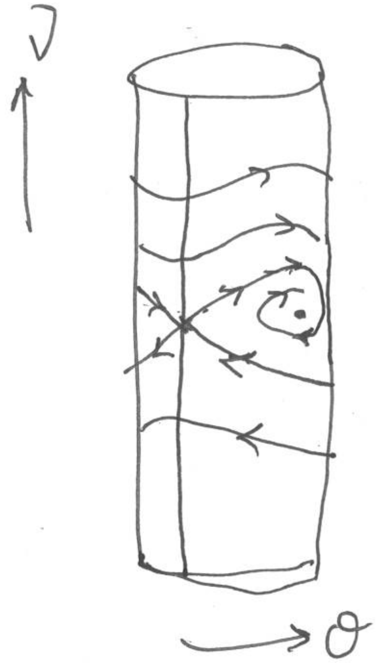
$$\Rightarrow \quad \frac{d}{d\tau}E = -b\dot{\theta}^2 \le 0,$$

i.e., for $b \neq 0$, E is a non-increasing function of τ.

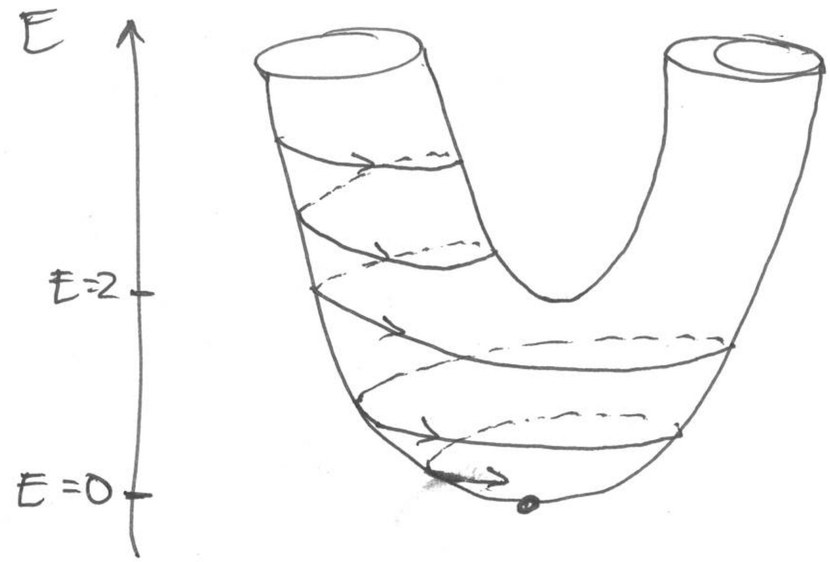
This means that phaseplane trajectories decay to regions of lower energy, ~~which~~ tending to the lowest energy trajectory, $E = 0$, i.e., tending to the fixed point $(\theta_f, \nu_f) = (0, 0)$



On the Cylindrical phase space, we have the following picture:



and after bending it, the U-tube looks like:



The U-tube clearly shows the decay of energy along trajectories of the Phase portrait, due to the friction term added to the pendulum equation.

This shows the power of the geometric point of view to analyse nonlinear equation.

Even that we can solve the equation, to interpret the solutions would be so difficult to deal with.

As a particular example, the frictionless pendulum can be solved in terms of the Jacobi Elliptic functions. Nevertheless, the solutions of the equation in terms of Elliptic functions can be useful:

0530~~20~~12010

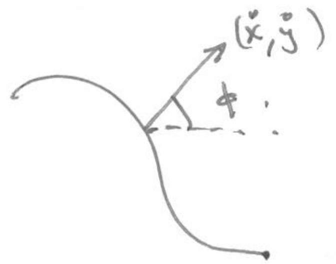
$\boxed{6.8}$ INDEX THEORY

Consider $\dot{x} = f(x)$ a smooth vector field.

and let C be a closed curve (not necessarily an orbit)

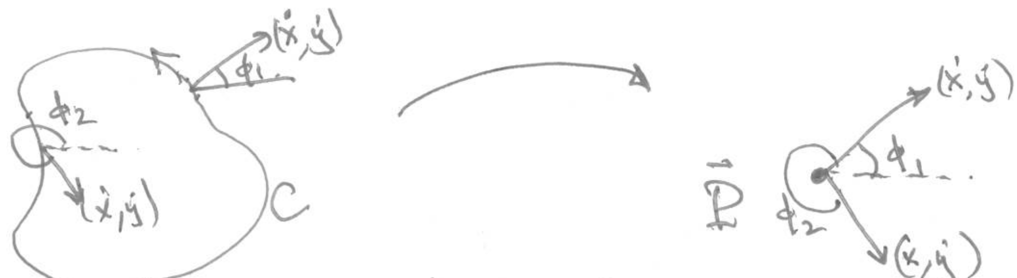
and C is a <u>simple curve</u> (it does <u>not</u> intersect to itself).

At each point $\bar{x} \in C$, the vector $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ makes an angle with the horizontal:



$$\phi = \begin{cases} \operatorname{Arctan}\left(\dfrac{\dot{y}}{\dot{x}}\right), & \text{if } \dot{x} > 0 \\ \operatorname{Arctan}\left(\dfrac{\dot{y}}{\dot{x}}\right) - \pi, & \text{if } \dot{x} < 0,\ \dot{y} < 0 \\ \operatorname{Arctan}\left(\dfrac{\dot{y}}{\dot{x}}\right) + \pi, & \text{if } \dot{x} < 0,\ \dot{y} \geq 0 \end{cases}$$

Over a close curve, the angle ϕ changes continuously so, after a turn, ϕ has moved a multiple of 2π.



This is to say, if we fix the arrow defined by $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ to fixed point P, and observe the rotation of $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ about P, as ~~the~~ we move ~~the~~ around C, $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ has been moved a number of times about P.

This multiple of 2π is called the index of the closed curve C:

$$I_C = \frac{1}{2\pi}[\phi]_C$$

where $[\phi]_C$ denotes the net change of ϕ about P, after one turn to C.

Properties of the index.

1. If C is deformed to C' without passing through a fixed point:

$$I_C = I_{C'}$$

2. If C does not enclose a fixed point, then:

$$I_C = 0.$$

3. If we reverse time $(t \longrightarrow -t)$, the index of a curve doesn't change:

$$I_C = I_{-C}$$

4. If C is a trajectory (solution) of the system $\dot{x} = f(x)$, then

$$I_C = +1$$

<u>Def</u> Let $\bar{x}_f$ be a fixed point of $\dot{x} = f(x)$. <u>The index of $\bar{x}_f$</u> is the index I_C, where C is a curve that encloses only to $\bar{x}_f$, and no other fixed pt.

<u>Example</u>: Index of a <u>saddle</u>: $I_S = -1$

Index of a <u>stable</u> <u>node</u>: $I_n = +1$

Index of an <u>unstable</u> <u>node</u>: $I_{un} = +1$

<u>Theorem: 6.8.1</u> If a curve C encloses n isolated fixed points $x_1^*, x_2^*, \dots, x_n^*$, and $I_1, I_2, \dots, I_n$ are its corresponding indices, then:

$$I_C = I_1 + I_2 + \dots + I_n$$

<u>Theorem</u> $\boxed{6.8.2}$ Any closed <u>orbit</u> in the <u>phase plane</u>. must enclose fixed points whose ~~sum~~ indices sum $+1$.

<u>Proof</u> B $\quad \sum_{k=1}^{n} I_k = +1$.

<u>Proof</u>: By property (4), $I_C = +1$. By previous thm: $I_C = \sum I_k$. Then: $\sum I_k = +1$ QED.

Exercise $\boxed{6.8.13}$ (a) and:

(b) Derive the integral formula:

$$I_C = \frac{1}{2\pi} \oint_C \frac{f\,dg - g\,df}{f^2 + g^2}.$$