PART TWO

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TWO-DIMENSIONAL FLOWS.

CHAPTER 5 LINEAR SYSTEMS.

One-dimensional systems are too restrictive. Either:

1) Stay there (fixed points)
2) tend to infinity
3) tend to fixed points.

In highest dimensions, they have more room to move. Instead of starting to study these type of systems in general, let's starts with the simplest of those: linear systems in two dimensions. They play an important role in the classification of fixed points in nonlinear systems.

5.1 Definitions and Examples.

A two-dimensional linear system is a system of the form:

$$\dot{x} = ax + by \qquad a, b, c, d \text{ are parameters.}$$
$$\dot{y} = cx + dy.$$

In matrix form.

$$\dot{\vec{x}} = A\vec{x}$$

where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

This system is linear in the sense that if x_1 and x_2 are solutions, then $\vec{x} = \alpha\vec{x}_1 + \beta\vec{x}_2$ is a solution with α, β constants.

Similarly, if $x=0$, then $\dot{x}=0$. I.e: $x_e=0$ is always a fixed point of $\dot{\bar{x}} = A\bar{x}$.

Evenmore, if

$$\det A \neq 0,$$

then:

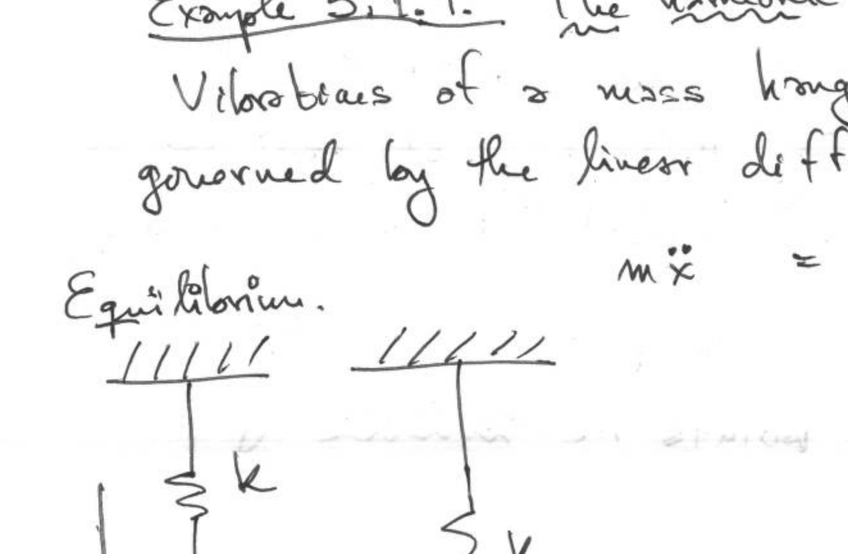
$$\vec{x}_e = 0 \text{ is always a fixed point.}$$

Example 5.1.1. The harmonic oscillator:

Vibrations of a mass hanging from a linear spring are governed by the linear differential equation:

$$m\ddot{x} = -kx. \quad \ldots\ldots (1)$$

Equilibrium.



m – mass of attached particle

k – spring's Hooke constant.

x – displacement of the particle from equilibrium

* Give a phase plane analysis.

We want to develop methods for deducing the behaviour of equations like (1), without actually solve them.

The state of the system is determined by the current position $x=x(t)$, and velocity $v=v(t)$.

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If both x and v are given, then eq (1) determines any future values of x and v.

We now write (1) as a system:

$$\dot{x} = v \qquad \text{i.e.} \quad \begin{pmatrix} x \\ v \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}.$$
$$\dot{v} = -\frac{k}{m} x$$

Define: $\omega^2 \equiv \dfrac{k}{m} \implies \begin{pmatrix} x \\ v \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}.$

To each point (x, v) corresponds a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$, i.e., this is a vector field on the phase-plane.

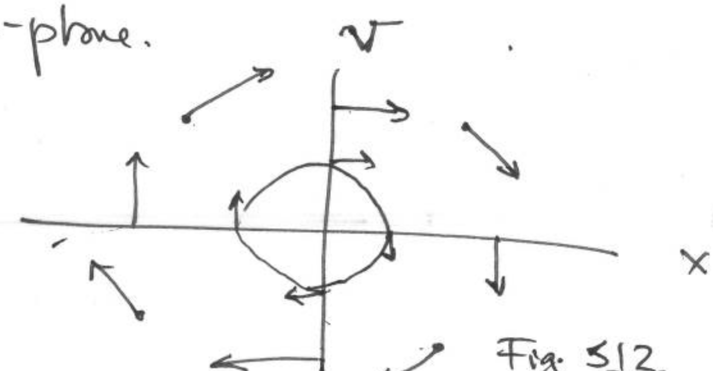

Fig. 5.1.2.

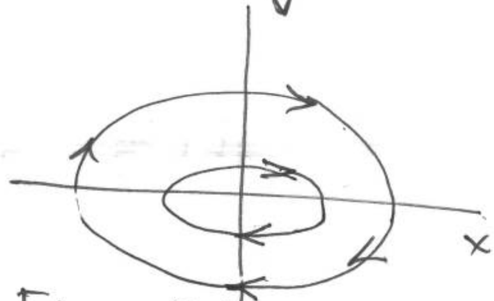

Figure 5.1.3.

Starting at a pt. (x_0, v_0), we observe the trajectory or "flow" of an imaginary particle on the phase plane.

Since $(x, v) = (0, 0)$ implies $(\dot{x}, \dot{v}) = (0, 0)$, then:

$\vec{x}_e = (0, 0)$ is an equilibrium solution or fixed point.

For the harmonic oscillator, we have closed orbits.

Figure 1.5.3. is called the phase portrait

Physical translation:

Fixed point $(0,0)$ $\longrightarrow$ Equilibrium position of the oscillator.

Closed orbits $\longrightarrow$ Oscillation of the particle.

Equation of the closed curves:

$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x.$$

i.e.

$$+\omega^2 x \dot{x} = \omega^2 x v$$
$$v\dot{v} = -\omega^2 v x. \quad \Rightarrow \quad \omega^2 x \dot{x} + v\dot{v} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\omega^2 x^2}{2} + \frac{v^2}{2}\right) = 0 \quad \Rightarrow \boxed{\omega^2 x^2 + v^2 = C} \quad C \geq 0.$$

The trajectories are ellipses. This is the conservation of energy

$$\frac{1}{2} m v^2 + \frac{1}{2} m \omega^2 x^2 = \frac{1}{2} m C$$

$$\boxed{T + V(x) = E}$$

with:
$$V(x) = \frac{1}{2} m \omega^2 x^2$$

i.e.
$$V(x) = \frac{1}{2} k x^2$$ is the potential energy for the harmonic oscillator.

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$\boxed{\text{Example 5.1.5}}$ Solve the linear system, and sketch the phase portrait, for: $\dot{x} = Ax$, with

$$A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with } a \in \mathbb{R}.$$

Show the different cases.

Soln: The system is:

$$\dot{x} = ax$$
$$\dot{y} = -y.$$

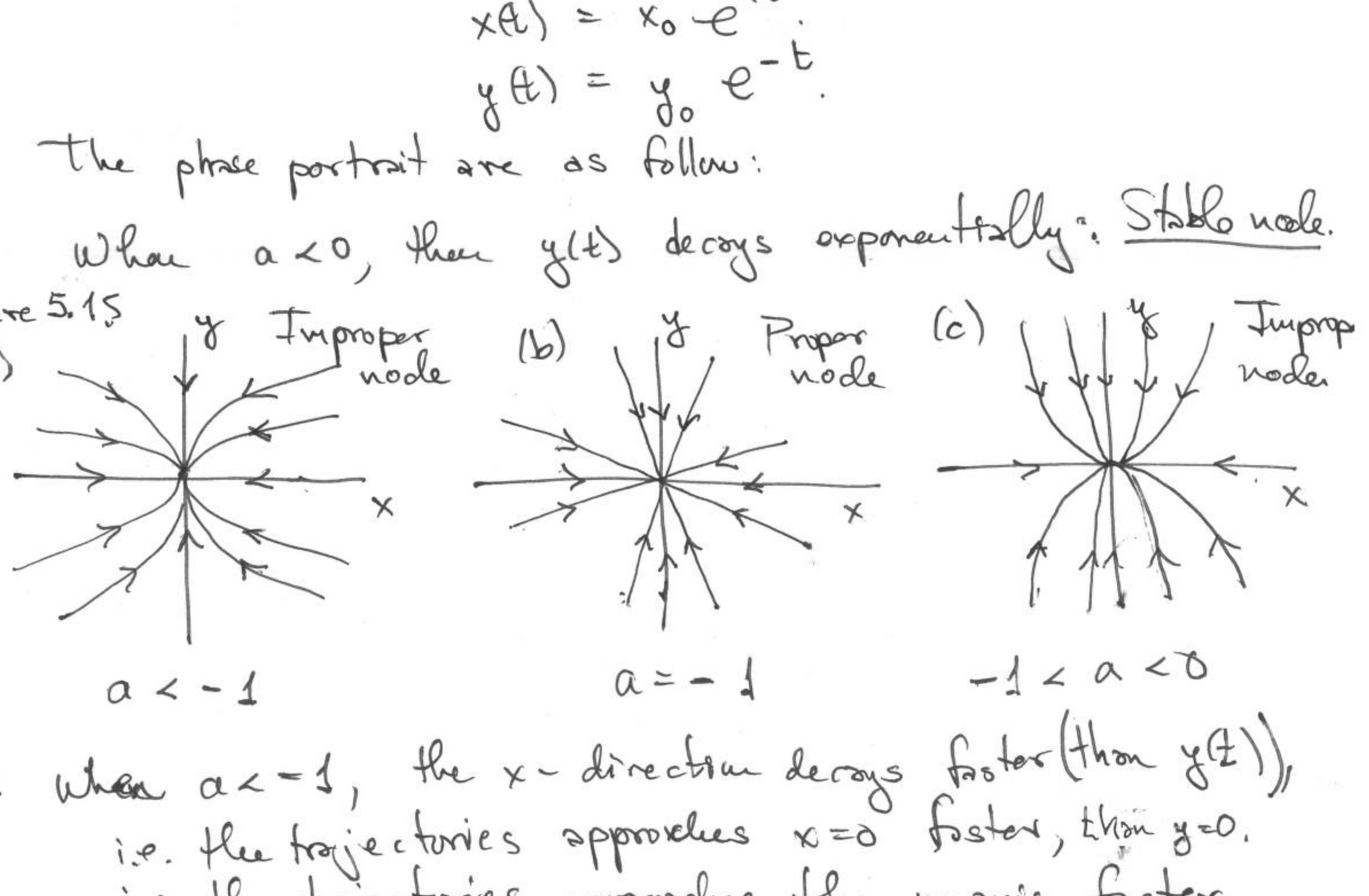
which is an uncoupled system. Solution:

$$x(t) = x_0 e^{at}$$
$$y(t) = y_0 e^{-t}.$$

The phase portrait are as follow:

When $a < 0$, then $y(t)$ decays exponentially: Stable node.

Figure 5.15



$a < -1$ $\qquad$ $a = -1$ $\qquad$ $-1 < a < 0$

* when $a < -1$, the x-direction decays faster (than $y(t)$),
i.e. the trajectories approaches $x=0$ faster, than $y=0$.
i.e. the trajectories approaches the y-axis faster.
and its approaches $\vec{x}_f = 0$. tangentially to y-axis.

If $t \to -\infty$, then the trajectories are parallel to the x-axis.
This is an improper node.

* The case $a = -1$ implies $\frac{y(t)}{x(t)} = \frac{y_0}{x_0}$ = constant;

Then; $y(t) = m\,x(t)$; all of the trajectories are straigth lines

This is a proper node. It is also called a symmetric node or a star.

(This is a very special case, $a = -1$, that is why it is called a proper node).

* When $-1 < a < 0$, we have again an improper node and again it is stable.

Now the trajectories approach $\vec{x}_e = 0$ tangentially to the x-axis.

Here, the $y(t)$ decays faster (than $x(t)$)

i.e., the trajectories approach to $y=0$ faster (than to $x=0$)

i.e., the trajectories approach $\vec{x}_f = 0$ tangentially to the x-axis

i.e., trajectories approach the x-axis faster (than the y-axis

If $t \longrightarrow -\infty$, then, trajectories become parallel to the y-axis.

* <u>When $a=0$.</u>

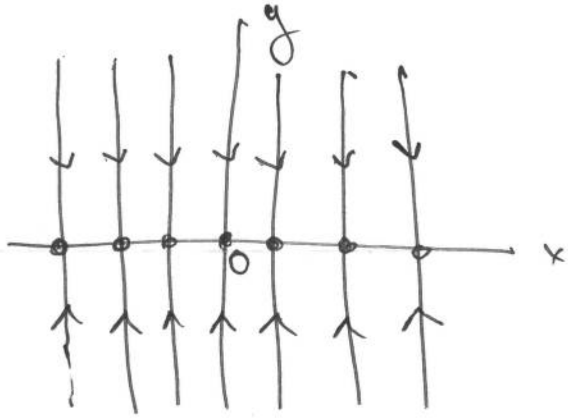
When $a=0$, something dramatic happens:

$$x(t) = x_0 \quad \text{for any } x_0 \in \mathbb{R}.$$

This means any point of the x-axis is a fixed point.

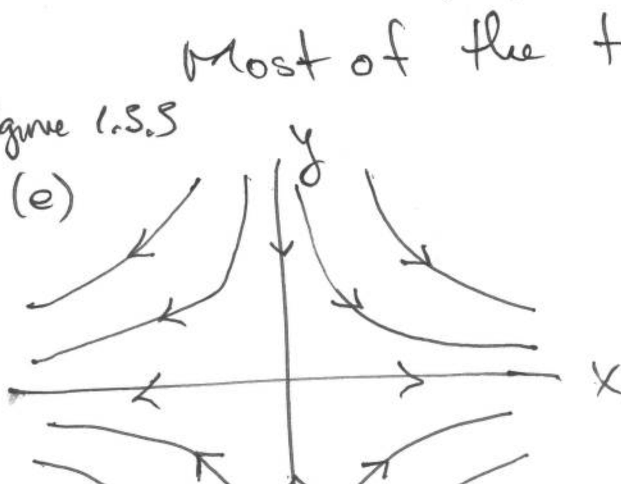
<u>Entire line</u> of <u>fixed pts.</u> along the x-axis.

Figure 5.1.5. (d)



* <u>When $a>0$</u>, the point $\bar{x}_f=0$ becomes <u>unstable</u>, due to the (exponential) growth in the x-direction. Most of the trajectories go away from $\bar{x}_f=0$ to ∞.

Figure 1.5.5 (e)



<u>Exception</u>: If $\bar{x}(0)=\bar{x}_0$ lies on the y-axis, and the solutions go straigth to the origin.

* If $t \longrightarrow +\infty$, trajectories are asymptotic to the x-axis
* If $t \longrightarrow -\infty$, trajectories are asymptotic to the y-axis.

These two comments will help to define the <u>stable</u> and <u>unstable manifolds</u>.

Here, $\bar{x}_f = 0$ is called a <u>saddle-point</u>.

The y-axis is called the <u>stable manifold</u> of the saddle point $\bar{x}_f = 0$.

<u>Def.</u> The <u>stable manifold</u> of a saddle point $\bar{x}_f$ is the set of all initial conditions $\bar{x}_0$ such that $\vec{x}(t) \longrightarrow \vec{x}_f$ as $t \longrightarrow +\infty$.

<u>Def:</u> The <u>unstable manifold</u> of a saddle point $\bar{x}_f$ is the set of all initial conditions $\bar{x}_0$ such that $\vec{x}(t) \longrightarrow \bar{x}_f$ as $t \longrightarrow -\infty$.

Here, the x-axis is the unstable manifold.

<u>Remark</u> (a) The trajectories approach the <u>unstable</u> manifold as $t \longrightarrow +\infty$.

(b) Now, as $t \longrightarrow -\infty$, the trajectories approach the <u>stable manifold</u>.

Stability Language:

* $\bar{x}_f = 0$ is an <u>attracting</u> fixed point if all trajectories starting near $\bar{x}_f$ approach $\bar{x}_f$ as $t \longrightarrow +\infty$, i.e., if $\bar{x}(t) \longrightarrow \bar{x}_f$ as $t \to \infty$, whenever $\bar{x}_0 = \bar{x}(0)$ is near $\bar{x}_f$.

* If all trajectories in the phase plane are attracted by $\bar{x}_f$, we say $\bar{x}_f$ is a <u>global attractor</u>.

* See figures 5.1.5 a, b, c in the text, or page =119= in these notes.

<u>Liapunov stability</u>.

This type of stability relates to the behaviour <u>for all t</u>, not only when $t \longrightarrow +\infty$. This is the <u>Liapunov</u> <u>Stability</u>.

<u>Definition</u>. <u>Liapunov stability</u>.

We say that a fixed point $\bar{x}_f$ is <u>Liapunov stable</u> if all trajectories that start sufficiently close to $\bar{x}_f$ will remain close to $\bar{x}_f$ for all t. (In figures 5.1.5 a, b, c, d the origin Liapunov stable, pages =119= and =121=, fig (d)).

* A fixed point can be Liapunov stable, but not attracting.

Example Fig. 5.1.5(d). The origin is Liapunov stable, but not an attracting fixed point: it is a "neutrally stable" fixed

Def: A neutrally stable fixed point is a Liapunov stable fixed point, but not an attracting fixed point.

Nearby trajectories of a neutrally stable fixed point are not attracted neither repelled from the fixed pt.

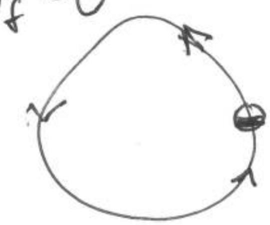
Example: A second example of a neutrally stable fixed point is the equilibrium point of the harmonic oscillator.

So far, we have seen two examples of Liapunov stable fixed pts., but not attracting.

Conversely, it is possible to find attracting fixed point which are not Liapunov, as in the following example.

Example: $\dot{\theta} = 1 - \cos\theta$.

Here: $\theta_f = 0$ is a fixed point, but it is not Liapunov, since there are trajectories closed to $\theta_f = 0$, but go away to come back later to $\theta_f = 0$



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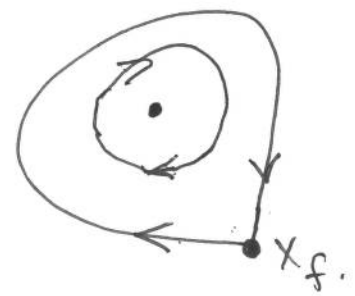
Ex: Another example corresponds the "homoclinic" orbits



In practice, the two types of stability occur simultanea

[Def] simultaneously. A fixed point $\bar{x}_f$ which is Liapunov stable and attracting, it is called stable or asymptotically stable.

[Def.] A fixed point $\bar{x}_f$ which is not Liapunov neither attracting is called an unstable fixed point.

[5.2] Classification of Linear Systems.

In the previous section, we had a diagonal matrix.
In the previous section, we had a diagonal matrix.
Here, we study the general case for a 2x2 matrix.

In the diagonal case: straight-line trajectories $\left\{ \begin{array}{l} x\text{-axis} \\ y\text{-axis} \end{array} \right.$

For the general case: A is a 2x2 matrix. We have the system:

$$\dot{\bar{x}} = A\bar{x} \qquad \text{-------------} (*)$$

Assume solutions of the form: $\bar{x}(t) = e^{\lambda t}\,\bar{v} \quad \text{----}(**)$

where $\vec{v}$ is a fixed vector (i.e. t - independent).
and $\vec{v} \neq 0$.

λ is the growth rate

$\vec{v}$ and λ have to be determined.

It follows from ~~sub~~ substituting (**) into (*) (previous page), we get the <u>eigenvalue problem</u>:

$$A\vec{v} = \lambda\vec{v}$$

λ - <u>eigenvalue</u> of the matrix A
$\vec{v}$ - <u>eigenvector</u> of the matrix A.

The eigenvalues λ solve the characteristic equation:

$$\det(A - \lambda I) = 0.$$

For a 2x2 matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic equation becomes:

$$\lambda^2 - \tau\lambda + \Delta = 0,$$

where

$\tau = a + d$ — Trace of $A = tr(A)$

$\Delta = ad - bc$ — Determinant of $A = \det(A)$

Then, the eigenvalues are:

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \quad ; \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Typical case $\lambda_1 \neq \lambda_2$.

Linear algebra shows there are two linearly independent corresponding eigenvectors $\bar{v}_1$ and $\bar{v}_1$.

Any initial condition can be written as:

$$\bar{x}_0 = c_1 \bar{v}_1 + c_2 \bar{v}_2.$$

This suggests to write the general solution $\bar{x}(t)$ to the system $\dot{\bar{x}} = A\bar{x}$ as:

$$\bar{x}(t) = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2.$$

This is solution to $\dot{\bar{x}} = A\bar{x}$ since:

$$\dot{\bar{x}} = c_1 e^{\lambda_1 t} \lambda_1 \bar{v}_1 + c_2 e^{\lambda_2 t} \lambda_2 \bar{v}_2 = c_1 e^{\lambda_1 t} A\bar{v}_1 + c_2 e^{\lambda_2 t} A\bar{v}_2$$

$$= A\left(c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2\right) = A\bar{x} \quad \checkmark$$

and also it satisfy the initial condition $\bar{x}(0) = x_0 = c_1\bar{v}_1 + c_2\bar{v}_2$. By the Existence and Uniqueness Thm, this is the only one solution.

Example. Consider the system:

$$\dot{x} = x + y \qquad \dot{y} = 4x - 2y.$$

with initial conditions $x(0) = 2$, $y(0) = -3$.

This is to say

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\bullet} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

The eigenvalues solve:

$$\lambda^2 + \lambda - 6 = 0 \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = -3 \end{cases}$$

The eigenvector $\vec{v} = (v_1, v_2)$ satisfy,

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

i.e. $(1-\lambda) v_1 + v_2 = 0$

$4 v_1 - (2+\lambda) v_2 = 0$ (These two eqs. are equivalents)

* For $\lambda_1 = 2$; $(1-2) v_1 + v_2 = 0$

$-v_1 + v_2 = 0$

Choos $v_1 = 1$, and $v_2 = 1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

* For $\lambda_2 = -3$ $(1+3) v_1 + v_2 = 0$.

$4 v_1 + v_2 = 0$

Choose $v_1 = 1$, $v_2 = -4 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

The general solution becomes.

$$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

It remains to find c_1 and c_2. At $t = 0$:

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Inverting the matrix:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Then: $$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence: $$\left.\begin{aligned} x(t) &= e^{2t} + e^{-3t} \\ y(t) &= e^{2t} - 4e^{-3t} \end{aligned}\right\}$$ is the solution to the initial value problem.

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Fortunately, we do not have to solve all the systems to be able to draw it's phase portrait. We only need to know the eigenvectors and eigenvalues.

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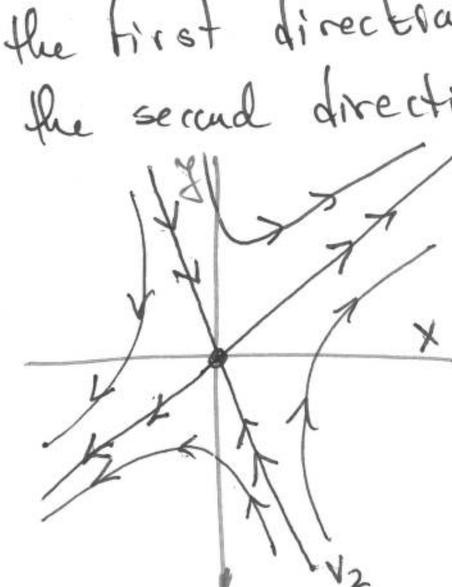
$\boxed{5.2.2}$ Example Draw the phase portrait for the system shown in 5.2.1.

Since $\lambda_1 = 2$, $\lambda_2 = -3$ we have a saddle point.

In the first direction, $\bar{v}_1$, we get exponential growth.
In the second direction, $\bar{v}_2$, we get exponential decay.



The origin is a saddle point.
Multiples $\bar{v}_1$ is the unstable manifold
Multiples $\vec{v}_2$ is the stable manifold

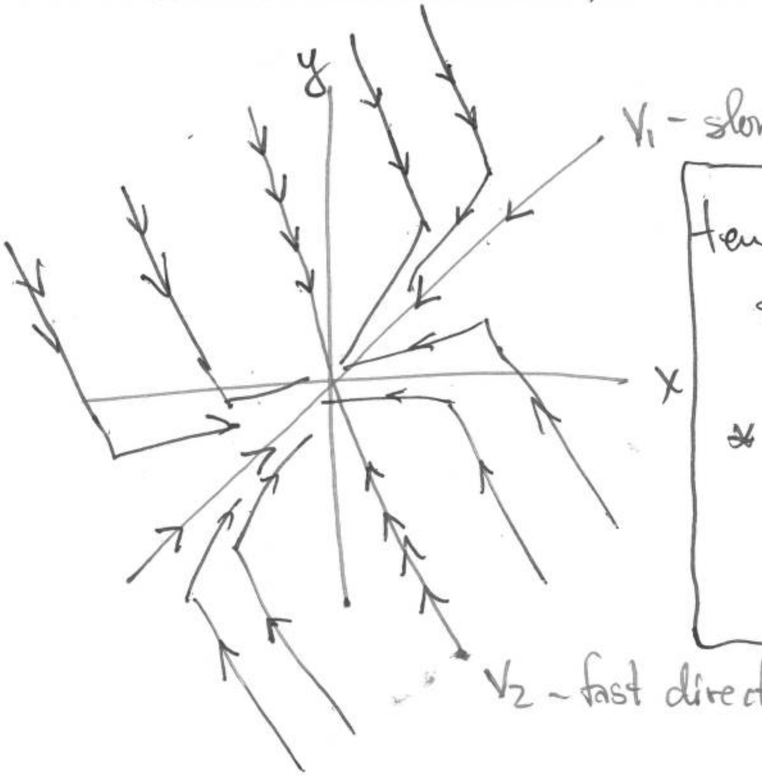
As in all saddle points, the trajectories are asymptotic to the <u>unstable manifold</u> when $t \longrightarrow +\infty$, and asymptotic to the <u>stable</u> manifold as $t \longrightarrow -\infty$.

Example: 5.2.3

Sketch the phase portrait for $\lambda_2 < \lambda_1 < 0$.

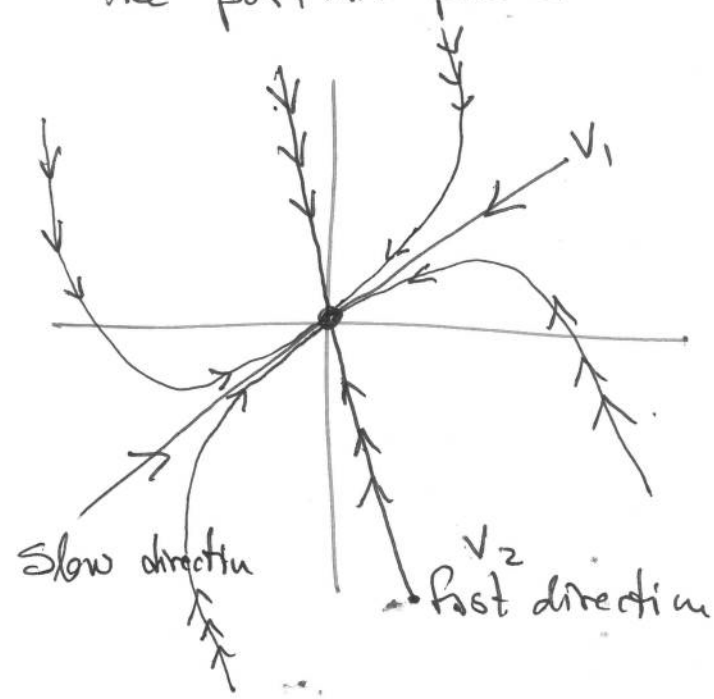
<u>Solution</u>: Let v_1 correspond to λ_1.
and v_2 correspond to λ_2.

* Since λ_1 and λ_2 are both negative, we have a node.
* Since $\lambda_1 \neq \lambda_2$, we have an <u>improper node</u>.
* Since $|\lambda_2| > |\lambda_1|$, then the line spanned by v_1 is the slow direction, and $\vec{v}_2$ defines the fast direction.



* Trajectories typically tend to $\vec{x}_f = 0$, tangentially to the slow direction.
* For $t \longrightarrow -\infty$, the trajectories tend to parallel lines to the fast direction.

The portrait phase looks like:



Remark:
(If we reverse the arrows or change the sign for the λ's:

$$0 < \lambda_1 < \lambda_2,$$

we obtain the typical phase portrait of an unstable (improper) node)

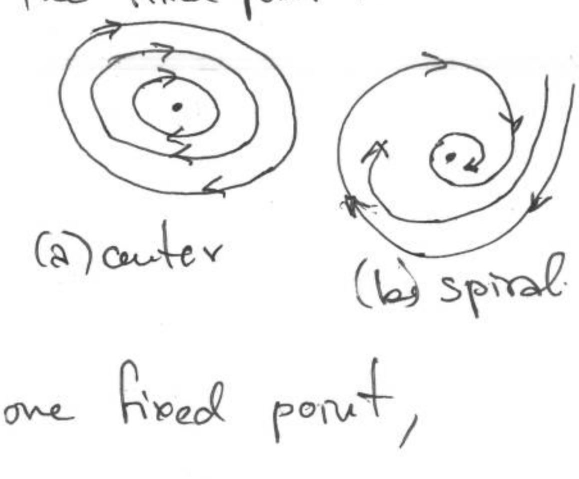
Example 5.2.4.

What happens if the eigenvalues are complex numbers?

If the eigenvalues are complex, the fixed point is

(a) either a center

(b) or a spiral



(a) center (b) spiral

* The harmonic oscillator has only one fixed point, which is a center.
* Note that the center are neutrally stable, since nearby trajectories are neither attracted nor repelled.
* A spiral occurs if a harmonic oscillator is damped, since it loses energy each turn around, not allowing to close the orbits.

Remember that

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right).$$

For complex eigenvalues:

$$\tau^2 - 4\Delta < 0$$

Let:

$$\lambda_{1,2} = \alpha \pm i\omega$$

with $\alpha = \frac{1}{2}\tau$ and $\omega = \frac{1}{2}\sqrt{|\tau^2 - 4\Delta|}$.

Since we have assumed, $\omega \neq 0$, we have two distinct eivalues, and two linearly independent eigenvectors, hence:

$$\bar{x}(t) = c_1 e^{\lambda_1 t}\,\bar{v}_1 + c_2 e^{\lambda_2 t}\,\bar{v}_2 \,.$$

Now, $c_1, c_2, \bar{v}_1, \bar{v}_2$ are now complex-valued, and Then, $\bar{x}(t)$ is a linear combination of $e^{(\alpha \pm i\omega)t}$, and so, a linear combination:

$$\bar{x}(t) = k_1 e^{\alpha t}\cos(\alpha t)\,\bar{u}_1 + k_2 e^{\alpha t}\sin(\alpha t)\,\bar{u}_1 ,$$

now written in real form.

* If $\alpha = \mathrm{Re}(\lambda_{1,2}) < 0$, we have exponential decay: <u>Stable spiral</u>
* If $\alpha = \mathrm{Re}(\lambda_{1,2}) > 0$, we have exponential growth: <u>Unstable spiral.</u>
* If the eigenvalues are purely imaginary, $\alpha = \mathrm{Re}(\lambda_{1,2}) = 0$, the solutions are <u>periodic</u> solutions, with period $T = \frac{2\pi}{\omega}$.

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It is easy to determine the orientation of the rotation. Just pick few points in the phase plane, and plot the vector field there.

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Example So far we've assumed the eigenvalues are distinct what happens if the eigenvalues are equal?

Solution: Here

$$\lambda \equiv \lambda_1 = \lambda_2 .$$

Two cases: (a) two linearly independent eigenvector

(b) one linearly independent eigenvector.

$\boxed{*}$ If there are two linearly independent eigenvectors, they span the plane. Therefore, every vector $x_0 \in \mathbb{R}^2$ is an eigenvector with eigenvalue λ. To prove this statement, let v_1, v_2 be the two linearly independent eivct. of A, then any vector x_0 $x_0 \in \mathbb{R}^2$ can be written as:

$$\bar{x}_0 = c_1 \bar{v}_1 + c_2 \bar{v}_2$$

Now:

$$A\bar{x}_0 = A(c_1\bar{v}_1 + c_2\bar{v}_2) = c_1 A\bar{v}_1 + c_2 A\bar{v}_2 = c_1\lambda\bar{v}_1 + c_2\lambda\bar{v}_2$$
$$= \lambda(c_1\bar{v}_1 + c_2\bar{v}_2) = \lambda\bar{x}_0 .$$

Then, any vector $x_0 \in \mathbb{R}$ is eigenvector of A:

$$A\bar{x}_0 = \lambda\bar{x}_0$$

This means that the matrix A stretches ~~#~~ any vector x_0 by an amount λ, then A must be a multiple of the identity I:

$$A = \lambda I .$$

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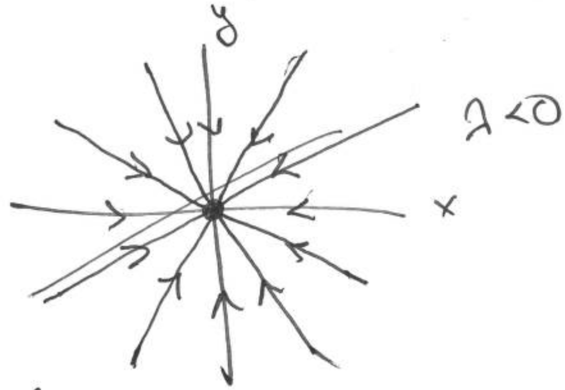
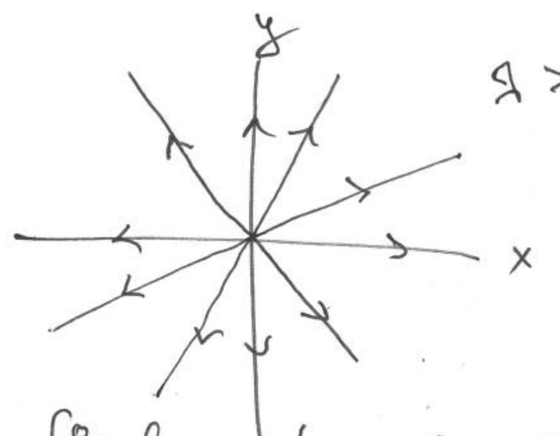
Then, if $\lambda \neq 0$, all solutions / trajectories go straight to the origin, and the fixed point (the origin) is a <u>star</u>, or a <u>proper node</u>.



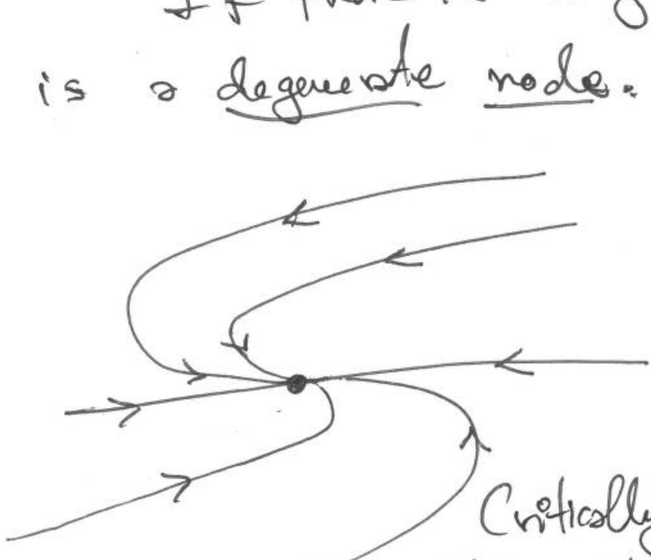


<u>If $\lambda = 0$</u>, the <u>full plane</u> are <u>fixed points</u>: $\dot{x} = 0$.

$\boxed{*}$ Only one ~~line~~ eigenvector: The eigenspace spanned, corresponding to λ, it is 1-dimensional.

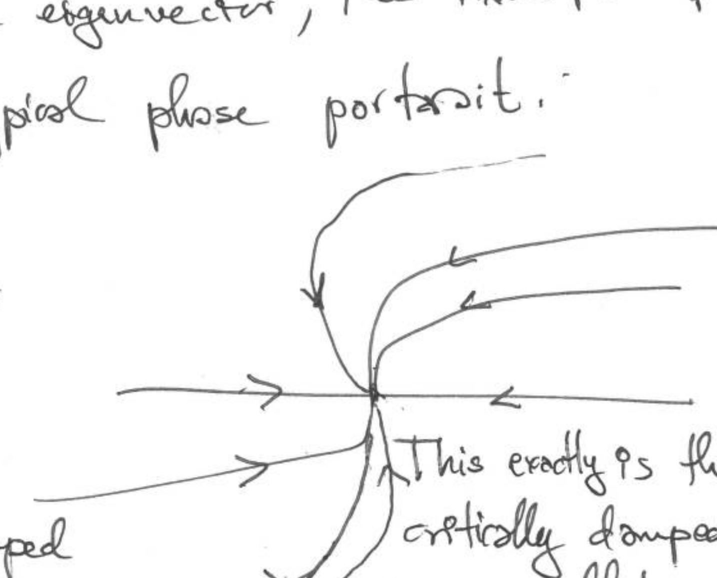
Any matrix of the form $\begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, with $b \neq 0$, has a 1-dim eigenspace. (Exercise 5.2.11)

If there is only one eigenvector, the fixed pt $\bar{x}_f = 0$, is a <u>degenerate</u> <u>node</u>. Typical phase portrait.



Critically damped "harmonic" oscillator

or

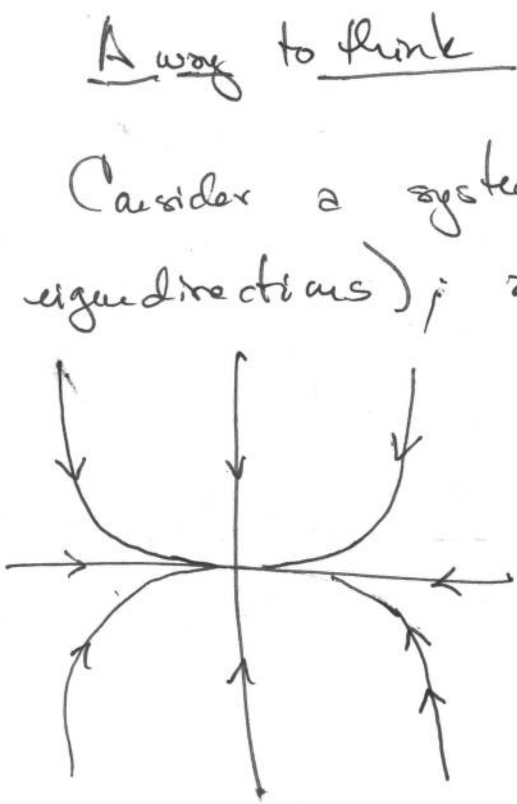


This exactly is the critically damped oscillator

Note that, when $t \to \infty$ or $t \to -\infty$, all trajectories go tangentially to the eigendirection

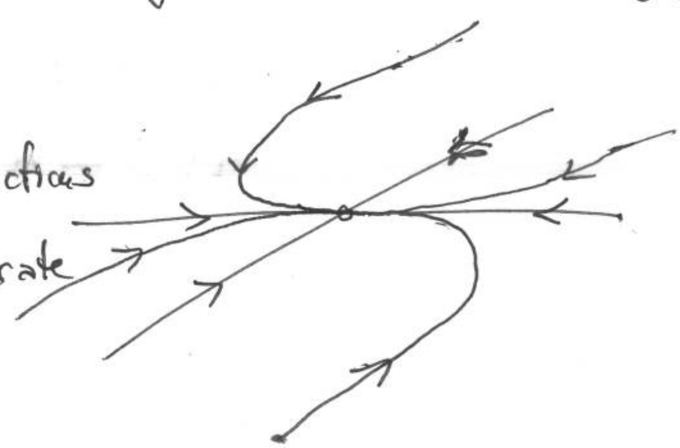
<u>A way</u> to <u>think</u> about the <u>degenerate eigendirection</u>:

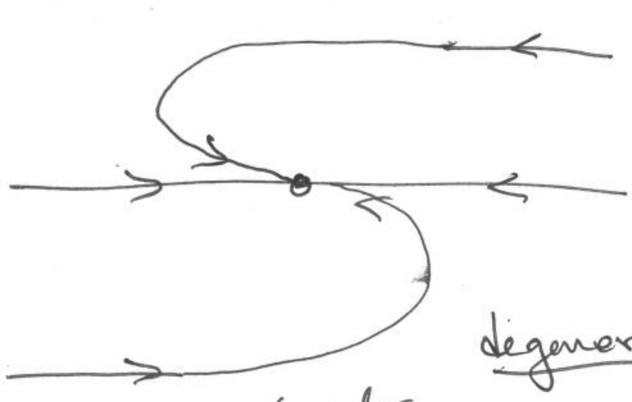
Consider a system with an <u>improper</u> <u>node</u> (and <u>two</u> eigendirections); and start to detune the parameters



so that one eigendirection (the vertical one, in this instant) starts to move to approach the other eigendirection (the horizontal one).

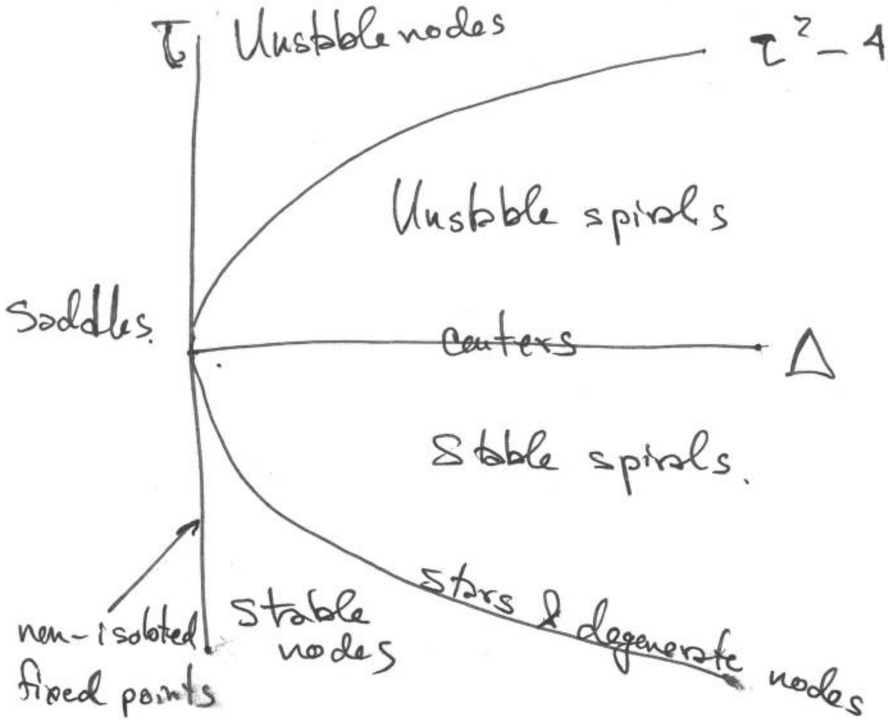
So that, when both eigendirections coincide, we obtain the degenerate case:





Also notice that the <u>degenerate</u> <u>node</u> is in the border line a <u>spiral</u> (underdamped oscillator) and a <u>node</u> (overdamped oscillator). The trajectories are trying to oscillate around the fixed point, but doesn't make it.

Classification of fixed points.



For the system:

$$\dot{x} = Ax.$$

we consider the matrix A, with determinant Δ and trace τ:

$$\Delta = \det(A) = \lambda_1 \lambda_2$$

$$\tau = tr(A) = \lambda_1 + \lambda_2.$$

The eigenvalues of A, λ_1, λ_2, solve:

$$\lambda^2 - \tau\lambda + \Delta = 0,$$

and are given by:

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}; \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}.$$

(a) If $\Delta < 0$, $\Rightarrow$ λ_1, λ_2 are real and have opposite signs

$\Rightarrow$ Saddle point.

(b) If $\Delta > 0$, then, either

(b.i) $\tau^2 - 4\Delta > 0$, improper nodes

or

(b.ii) $\tau^2 - 4\Delta < 0$, spirals and centers.

or (b.iii) $\tau^2 - 4\Delta = 0$, stars and degenerate nodes (b.i) $\tau^2 - 4\Delta > 0$: We are to the <u>left</u> of the <u>parabola</u>

We have two real eigenvalues, and distinct.

Then, we have <u>nodes</u>. (improper nodes).

* If $\tau < 0$, then $\lambda_2 < \lambda_1 < 0$

  then the <u>nodes</u> are <u>stable</u>

* If $\tau > 0$, then $0 < \lambda_2 < \lambda_1$.

  then the <u>nodes</u> are unstable.

(Note that $0 < \lambda_1 - \lambda_2$, always).

(b.ii) $\tau^2 - 4\Delta < 0$. We are to the <u>rigth</u> of the parabola

We have complex eigenvalues: <u>spirals</u> or <u>centers</u>.

* If $\tau < 0$, then $Re(\lambda_1) = Re(\lambda_2) < 0$.

  then the <u>spirals</u> are <u>stable</u>

* If $\tau > 0$, then $Re(\lambda_1) = Re(\lambda_2) > 0$,

  then, the <u>spirals</u> are <u>unstable</u>

* If $\tau = 0$, then $Re(\lambda_1) = Re(\lambda_2) = 0$,

  then, we have centers, which

  are <u>neutrally</u> <u>stable</u>

(b.iii) $\tau^2 - 4\Delta = 0$. We are <u>on</u> the parabola.

This case corresponds to repeated eigenvalues. $\lambda_1 = \lambda_2 = \frac{\tau}{2}$

Then, either we have:

* <u>Stars</u> (or <u>proper nodes</u>), when we have two eigendirections

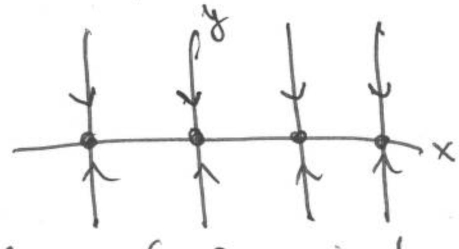
* <u>degenerate nodes</u>, when we have only one eigendirection.

Again, they are <u>stable</u> if $\lambda_1 = \lambda_2 = \frac{\tau}{2} < 0$, and <u>unstable</u> if $\lambda_1 = \lambda_2 = \frac{\tau}{2} > 0$.

(c) <u>$\Delta = 0$</u>. Here, at least one <u>eigenvalue</u> is <u>zero</u>

since $\Delta = \lambda_1 \lambda_2 = 0$.

Then, the origin is not an isolated fixed point:



or

Full line of fixed pts. Full plane of fixed pts.

---

* Saddle points, nodes and spirals are the most common type of fixed points. (larger open regions in (Δ, τ)).
* Centers, stars, degenerate nodes, and non-isolated fixed pts. are <u>not</u> common fixed pts. These are <u>borderline</u> <u>cases</u>.

Among the most important border line cases, the most important are the Centers, since they appear in conservative systems, such as in the harmonic oscillator, and frictionless mechanical systems.

Example $\boxed{5.26.}$: Classify the fixed point $\bar{x}_f = 0$ of the system $\dot{\bar{x}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \bar{x}$.

Soln: Since: $\det(A) = 4 - 2 \cdot 3 = -2$

$$tr(A) = 5$$

we observe $\Delta = \det(A) < 0$, then $\bar{x}_f = 0$ is a saddle.

Example $\boxed{5.27.}$ Classify the fixed point $\bar{x}_f = 0$, with $\dot{x} = Ax$.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

Soln: Here $\tau = 2 + 4 = 6$

$$\Delta = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5$$

Since $\tau > 0$, we have an unstable type of thing.

and

Since $\Delta > 0$, we have either a node, or a spiral or a center.

Now $\tau^2 - 4\Delta = 6^2 - 4 \cdot 5 = 36 - 20 = 16 > 0$.

We have an unstable improper node.

Remark We have the following nomenclature for nodes

Nodes

- Degenerate : 
  - * Repeated eivals $\lambda_1 = \lambda_2 = \lambda$
  - * Only one eigendirection
- Non-degenerate (two-eigendirections)
  - * Stars, or proper nodes
    $\lambda_1 = \lambda_2 \equiv \lambda$ (repeated eigenvalues)
  - * Nodes, or improper nodes.
    $\lambda_1 \neq \lambda_2$.

All of them can be stable or unstable, depending on the sign of τ.

Chapter 6 Phase Plane. 0129032010.

We start the study of nonlinear systems.

(i) General properties

(ii) Classification of fixed points, via linear analysis (Chap. 5)

(iii) Examples: Biology and Physics

(iv) Index theory

This chapter: fixed points

Next two chapters: closed orbits & bifurcations. in 2-D

6.1 Phase Portraits

Consider the vector field in the plane:

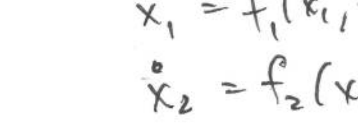
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Where f_1 and f_2 are given. In compact form:

$$\dot{\mathbf{X}} = f(\bar{x})$$

where: $\bar{x} = (x_1, x_2)^t$, $f = \left(f_1(\bar{x}), f_2(\bar{x})\right)^t$.

Giving a point $\bar{x}_0$ in the phase plane, and flowing the vector field, we have a trajectory $\bar{x}(t)$ through the phase plane.



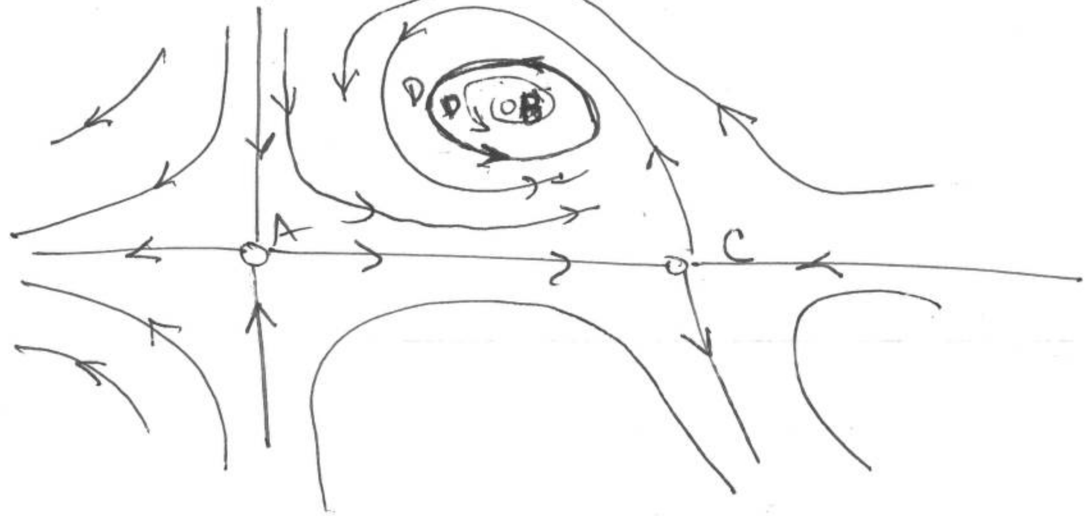
The full phase plane is full of trajectories, since each point corresponds to an initial condition.

For nonlinear systems, it is very hard to find analytical solutions. In the case they are known, it is hard to figure out how they look like. That is why we use.

geometric methods to determine the qualitative behaviour of the solutions of the system.

Goal: to find the system's phase portrait from the properties of $f(\bar{x})$.

Example of how a phase portrait looks like:



The most remarkable properties of almost any phase plane:

1. Fixed points. (Points A, B, C). $\dot{\bar{x}} = 0$, which implies: $f(\bar{x}) = 0$

2. Closed orbits (Trajectory D) Periodic orbits $x(t)$, such that $\bar{x}(t+T) = \bar{x}(t)$, for some $T > 0$.

3. The type of trajectories near fixed points and orbits.

4. Stability and Instability of closed orbits and fixed points. Points A, B, C are unstable. Trajectory D is stable

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Numerics of Phase portraits

Runge-Kutta is the standard.

Direction field. At any point $(x_1, x_2)^t$, we assign a velocity $(\dot{x}_1, \dot{x}_2)$, which describes the direction of the field at the point of consideration.

Example 6.1.1. $\dot{x} = x + e^{-y}$
$$\dot{y} = -y.$$

Fixed points: Constant solutions, stationary solutions, equilibrium solutions.

$$\begin{matrix}\dot{x}=0\\ \dot{y}=0\end{matrix} \Rightarrow \left.\begin{matrix} x+e^{-y}=0 \\ y=0\end{matrix}\right\} \Rightarrow \begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

is the fixed pt. (only one)

Note that: $y(t) = y(0)e^{-t}$ (y is decoupled).

$$\Rightarrow y(t) \xrightarrow[t\to\infty]{} 0,$$

Then, for large values of t:

$$\dot{x} = x + 1 \quad \left(\Rightarrow x(t) = (x_0+1)e^{t} - 1\right)$$

Then, the solutions grow exponentially: this suggests that the fixed point $(-1,0)^t$ is unstable.

Restrict solutions to the x-axis: $y_0 = 0$, then $y(t) \equiv 0$ $\forall t$.

Then, for the x-axis $\dot{x} = x+1$; then, the fixed pt. is unstable.

Nullclines Curves in the phase plane such that:

either $\dot{x} = 0$ or $\dot{y} = 0$.

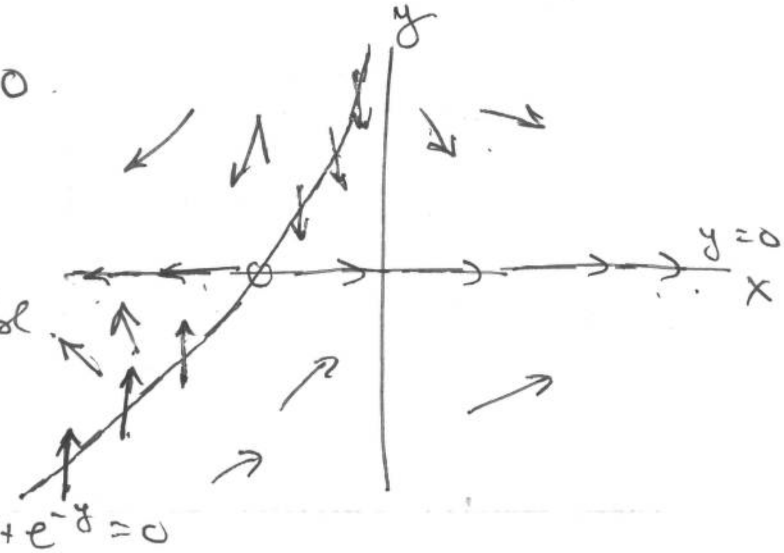
The intersection of nullclines are fixed points.

Null cline $\dot{x}=0$: vector field is vertical

Nullcline $\dot{y}=0$: vector field is horizontal

Nullclines: $x+e^{-y}=0$

$y=0$.



For $y=0$: $\dot{y}=0$; horizontal

$\dot{x}=x+1>0$, if $x\geq -1$.

$\dot{x}=x+1<0$, if $x<-1$.

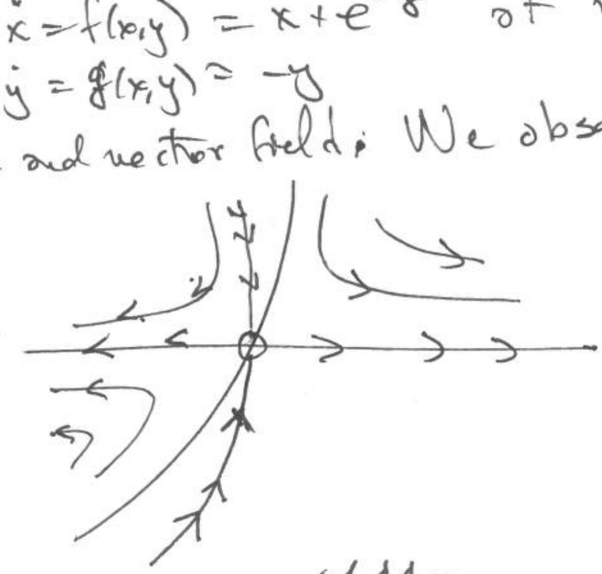
For $x+e^{-y}=0$. $\dot{x}=0$, vertical.

$\dot{y}=-y>0$, for $y<0$

$\dot{y}=-y<0$, for $y>0$.

Nullclines also provide a partition of the phase plane, with behavior of the vector field on those sectors, and also provides the vector field.

Evaluate $\dot{x}=f(x,y)=x+e^{-y}$ at many points to plot
$\dot{y}=g(x,y)=-y$
the slope and vector field. We observe the phase portrait look like:



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6.2 Existence, Uniqueness and Topological Consequences.

So far, we don't have any guarantee that the nonlinear system $\dot{x} = f(x)$ has a solution.

We can generalize the Theorem to n-dimensional systems

Existence and Uniqueness Theorem. Let the initial value problem

$$\dot{\vec{x}} = f(x)$$
$$\vec{x}(0) = \vec{x}_0$$

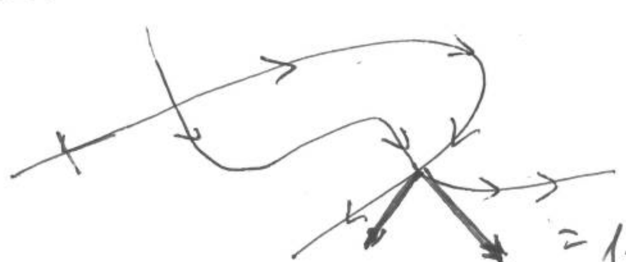
Suppose that $f(x)$ is continuous and all its partial derivatives $\partial f_i/\partial x_j$; $i,j = 1,\dots,n$ are continuous in some open connected set $D \subset \mathbb{R}^n$. For $x_0 \in D$, then, the Initial Value Problem has a solution $\bar{x}(t)$ on some interval $(-\tau, \tau)$, about $t=0$, and the solution is unique.

Existence and uniqueness of solutions are guaranteed if $f(x)$ and its derivatives are continuous.

We assume all the vector fields are smooth enough to guarantee existence and uniqueness.

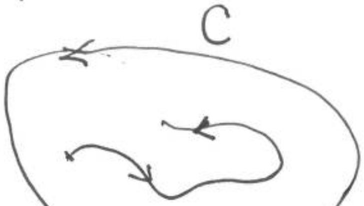
This theorem has an important consequence (corollary):

different trajectories never intersect. If two trajectories intersect, then there would be two solutions for the same point (initial value). This would violate the uniqueness part. Intuitively, a trajectory cannot move into different directions at once.



In 2-dim phase-spaces (as opposed to higher-dimensional spaces), these results have topological consequences.

Suppose there is a closed trajectory or orbit C in the phase plane. Then, any trajectory with initial condition inside the orbit C, stays inside C.



If there are fixed points, the trajectory eventually will approach to it. (stable fixed pt)

If there aren't fixed pts, then, the trajectory cannot roam forever. Then, it should go away from any fixed point, and should approach the orbit C.

For vector fields in the plane, the Poincaré-Bendixson theorem states that if a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must eventually approach the closed orbit. (Section 7.3 ahead, for this theorem)

6.3 Fixed points and Linearization.

We hope to have a description of the phase portrait of a nonlinear systems via the linearization near the fixed points.

Linearization.

Consider:
$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y),$$

and let (x_f, y_f) a fixed point: $\frac{d}{dt}x_f = 0$

an equilibrium solution or stationary solution: $\frac{d}{dt}y_f = 0$.

Then:
$$f(x_f, y_f) = 0$$
$$g(x_f, y_f) = 0$$

Let u, v such that $x = x_f + u$, $y = y_f + v$ is near the fixed point (x_f, y_f), ie., u, v are "small" (in some norm).

Notice that $u = u(t)$ and $v = v(t)$.

We can describe the behavior of $u(t)$ and $v(t)$ for small values of u and v, as follows:

$$\frac{du}{dt} = \frac{d}{dt}(x - x_f) \underset{\uparrow\, x_f \text{ is fixed pt.}}{=} \frac{dx}{dt} = f(x,y) = f(x_f + u, y_f + v) \overset{\text{Taylor's expansion}}{=}$$

$$= f(x_f, y_f) + u\frac{\partial f}{\partial x}(x_f, y_f) + v\frac{\partial f}{\partial y}(x_f, y_f) + O(u^2, v^2, uv)$$

$$\underset{(x_f, y_f) \text{ is fixed pt}}{=} u\frac{\partial f}{\partial x}(x_f, y_f) + v\frac{\partial f}{\partial y}(x_f, y_f) + O(u^2, v^2, uv)$$

Similarly, for $\dot{v}$.

If we drop the quadratic terms (assuming they are small), we obtain the following system (linear!) for u and v:

$$\dot{u} = \frac{\partial f}{\partial x}(x_f, y_f)u + \frac{\partial f}{\partial y}(x_f, y_f)v$$

$$\dot{v} = \frac{\partial g}{\partial x}(x_f, y_f)u + \frac{\partial g}{\partial y}(x_f, y_f)v$$

or, in matrix form: Jacobian matrix.

$$\begin{pmatrix} u \\ v \end{pmatrix}^{\cdot} = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(x_f, y_f)} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Notice that the partial derivatives are evaluated at the fixed point (x_f, y_f). Then, they are numbers. Then, we have a linear system with constant coefficients and we know how to study it! (Section 5.2).

The effect of small nonlinear terms

The text has few good questions:

1) Is it really safe to neglect quadratic terms?
2) Does the linear system give a good qualitative picture of the phase portrait corresponding to the nonlinear system?

Answer: Yes.

As long as the fixed point for the linearized system is not one of the borderline cases (in section 5.2).

This is to say:

If the linear system predicts a saddle, a node or a spiral, the fixed point really is a saddle, a node or a spiral in the nonlinear system.

The borderline cases (centers, degenerate nodes, stars, and non-isolated fixed points) are very delicate, and they can be altered by nonlinear terms.

Example 6.3.1. Consider the system:

$$\dot{x} = -x + x^3$$
$$\dot{y} = -2y$$

Find its fixed points, classify them in terms of linearization, and sketch its phase portrait. for the full nonlinear system.

Soln: Fixed points: $\begin{matrix}\dot{x}=0\\ \dot{y}=0\end{matrix}$, i.e.,

$$\left.\begin{matrix}-x+x^3=0\\ -2y=0\end{matrix}\right\} \Rightarrow \left.\begin{matrix}-x(1-x^2)=0\\ y=0\end{matrix}\right\} \begin{matrix}x=0,+1,-1\\ y=0.\end{matrix}$$

Then, we have 3 fixed pts:

$$(x_0,y_0)=(0,0),\ (x_1,y_1)=(1,0) \text{ and } (x_2,y_2)=(-1,0).$$

The Jacobian matrix is given by:

$$J(x,y)=\begin{pmatrix}-1+3x^2 & 0\\ 0 & -2\end{pmatrix}$$

Evaluate now J at the fixed points.

At $(x_0, y_0) = (0,0)$:

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

$\Delta = 2$, and $\tau = -3$. This means a stable something

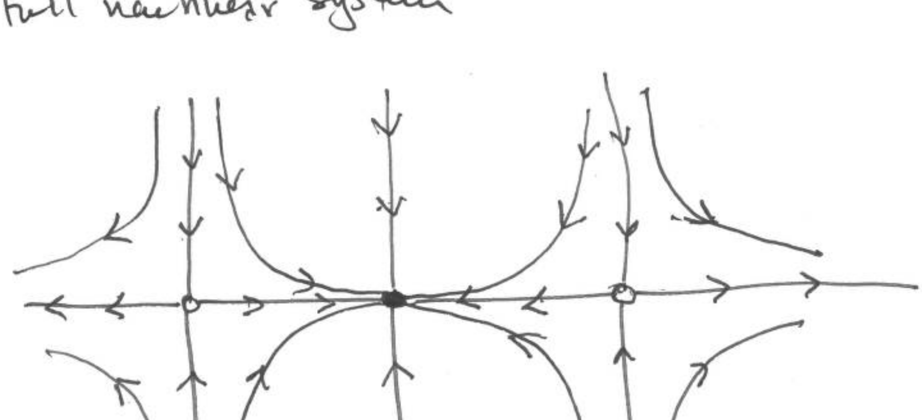
Since the eigenvalues are -1 and -2, means this is a stable (improper) node.

At $(x_1, y_1) = (1,0)$:
and $(x_2, y_2) = (-1,0)$

$$J(\pm 1, 0) = \begin{pmatrix} +2 & 0 \\ 0 & -2 \end{pmatrix}$$

This is a saddle, since $\Delta = -4$ or simply because its ei.val's are $+2$ and -2

Since these are not border-line fixed pts for the linear system we can state without hesitation that they behave similarly in the full nonlinear system



* The system is uncoupled, then orthogonal.
* In the y-direction, every trajectory decay to $y=0$, i.e. to the x-axis.
* The vertical lines $x=0, +1, -1$ are invariant, since $\dot{x}=0$ along them.
* In the x-direction, every trajectory is attracted to $x=0$, and repelled from: $x=+1, -1$.
* the horizontal line $y=0$ is also invariant, since $\dot{y}=0$ along it.
* Note the symmetry in the nonlinear system:

$$x \longrightarrow -x$$
$$y \longrightarrow -y.$$

Then, the phase portrait has these geometric symmetries.

The next example shows how small nonlinearities can perturb a borderline case, a center in this instance.

Example 6.3.2: Consider the system:

$$\dot{x} = -y + ax(x^2+y^2)$$
$$\dot{y} = +x + ay(x^2+y^2)$$

where a is a real parameter. Show that the linearization predicts a center at the origin, while the nonlinear system predicts a spiral, stable if $a<0$, unstable if $a>0$.

Solution: The Jacobian matrix is.

$$J(x,y) = \begin{pmatrix} a(3x^2+y^2) & -1+2axy \\ 1+2ayx & a(x^2+3y^2) \end{pmatrix}$$

At the origin:

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then the linear system is:

$$\dot{u} = -v$$
$$\dot{v} = u.$$

Remark: Note that the linearization about the origin is equivalent to drop the nonlinear terms in the original nonlinear system. End of the remark.

Note that the linear system has

$$\tau = 0, \quad \Delta = +1.$$

and this is a center, according to linear analysis

To analyze the nonlinear system, use polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

with $r \geq 0$ and $\theta \in [-\pi, \pi]$

Now from: $x^2+y^2 = r^2$, it follows that:

$$r\dot{r} = x\dot{x} + y\dot{y}$$

Hence

$$r\dot{r} = x(-y + ax(x^2+y^2)) + y(x + ay(x^2+y^2))$$

$$= -xy + ax^2(x^2+y^2) + yx + ay^2(x^2+y^2)$$

$$= a(x^2+y^2)(x^2+y^2)$$

$$= a(x^2+y^2)^2 = ar^4$$

$$\Rightarrow \quad \dot{r} = ar^3.$$

Now notice that. $\frac{y}{x} = \tan\theta$. Hence, computing the derivati[ve] $\frac{d}{dt}$ on both sides of the equation:

$$\frac{x\dot{y} - y\dot{x}}{x^2} = (1 + \tan^2\theta)\,\dot{\theta}$$

i.e.

$$\frac{x\dot{y} - y\dot{x}}{x^2} = \left(1 + \frac{y^2}{x^2}\right)\dot{\theta}$$

i.e.

$$\frac{x\dot{y} - y\dot{x}}{x^2} = \left(\frac{x^2+y^2}{x^2}\right)\dot{\theta} \Rightarrow \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2+y^2}$$

i.e.

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

Now, using the dynamical system.

$$\dot{\theta} = \frac{x(x + ay(x^2+y^2)) - y(-y + ax(x^2+y^2))}{r^2}$$

$$= \frac{x^2 + axy(x^2+y^2) + y^2 - ayx(x^2+y^2)}{r^2}$$

$$= \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2}$$

Hence $\dot{\theta} = 1$.

The full nonlinear system in polar coordinates becomes

$$\dot{r} = a r^3$$
$$\dot{\theta} = 1$$

Notice the solutions are always ~~spiraling~~ rotating in the counterclockwise direction, at a constant rate $\dot{\theta} = 1$.

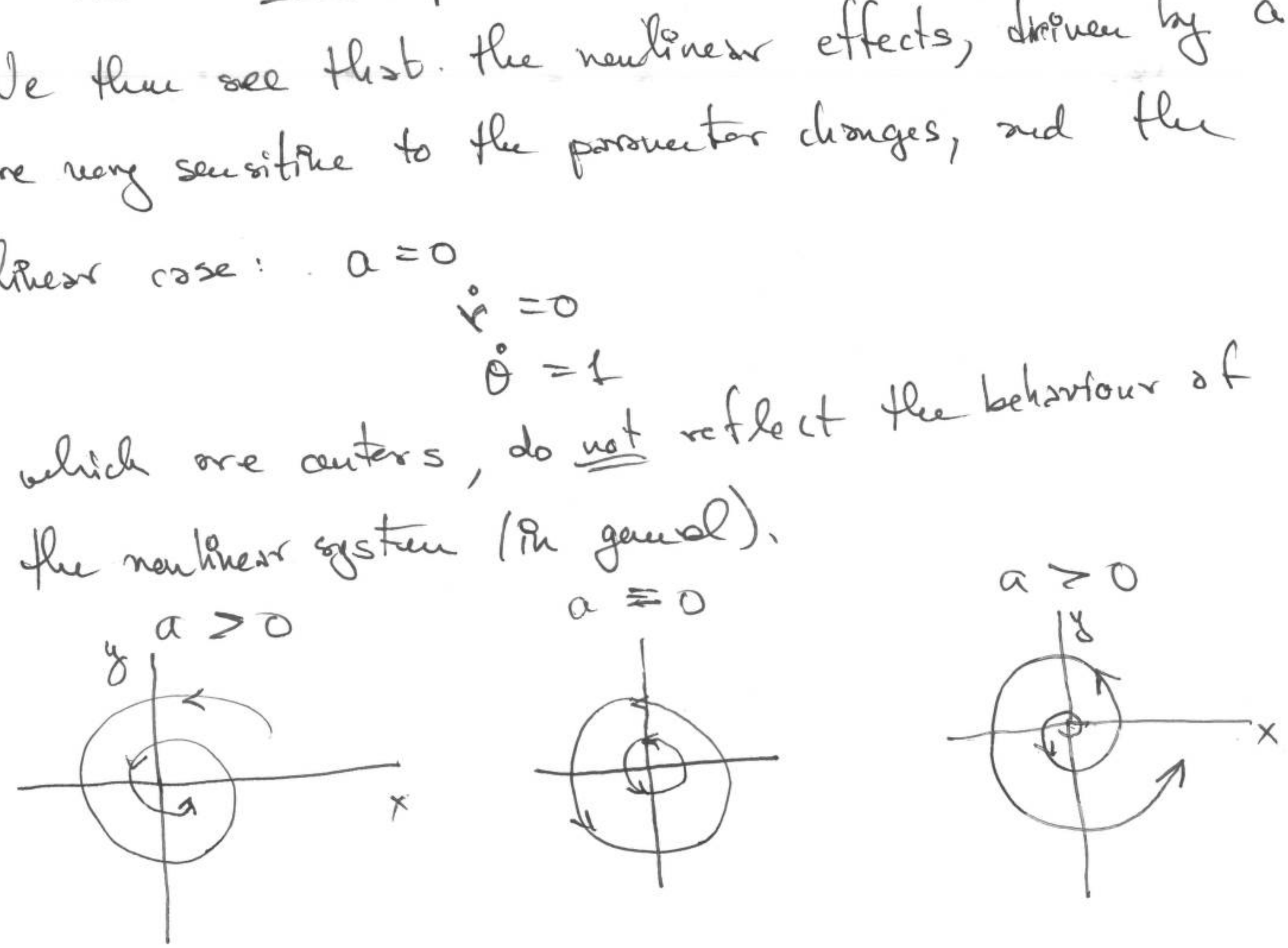
Similarly, if $a > 0$, $\dot{r} > 0$, then $r \nearrow$. This means solutions are unstable spirals.

If $a < 0$, $\dot{r} < 0$, then $r \downarrow 0$, and the solutions are stable spirals.

We then see that the nonlinear effects, driven by a, are very sensitive to the parameter changes, and the

linear case: $a = 0$
$$\dot{r} = 0$$
$$\dot{\theta} = 1$$

which are centers, do not reflect the behaviour of the nonlinear system (in general).



* Stars and degenerate nodes can be altered by small nonlinearities. However, they preserve its stability.

* For centers, after small nonlinear terms, its stability can change

If we are only interested in the stability of the fixed point, we can restrict to the following cases:

Robust cases

* Repellers (or sources): both eigenvalues have positive real part. ($\tau > 0$, $\Delta > 0$).
* Attractors (or sinks): both eigenvalues have negative real part. ($\tau < 0$, $\Delta > 0$).
* Saddles: eigenvalues are real and have opposite signs. ($\Delta < 0$)

Marginal cases:

* Centers: both eigenvalues are purely imaginary. ($\tau = 0$, $\Delta > 0$).
* Higher order and non-isolated fixed points: at least one eigenvalue is zero ($\Delta = 0$).

From the stability point of view, the marginal cases are those for which $Re(\lambda) = 0$.

Hyperbolic Fixed Points, Topological Equivalence and Structural Stability.

If $Re(\lambda) \neq 0$ for both eivals (eigenvalues) of a linearization about a fixed point, that fixed point is called hyperbolic. (this term does not have to deal with "saddle points").

Hyperbolic fixed points are strong: their stability does not change by small nonlinear terms.

Non-hyperbolic fixed points are fragile: their structure can change by nonlinear terms (independently of its size).

On the line, we had the case $f'(x_f) \neq 0$, to keep the linear structure onto the nonlinear system. Here, we have $\det(J(x_f, y_f)) \neq 0$, or $tr(J(x_f, y_f)) \neq 0$ (this is to say, $Re(\lambda) \neq 0$). for two dimensional systems.

For n-dimensional systems, a fixed point is hyperbolic if $Re(\lambda_j) \neq 0$, for $j = 1, 2, \ldots, n$.

Hartman-Grobman Theorem.

The local phase portrait near a hyperbolic fixed point is "topologically" equivalent to the phase portrait of the linearization. The stability type of the fixed point is accurately captured by the linearization.

Here, topologically equivalent means that there is a homeomorphism (a continuous function with a continuous inverse) that maps one local phase portrait onto the other. In such case, trajectories are also mapped into trajectories and the sense of time is preserved.

Intuitively, topologically equivalent means that we can deform the one phase portrait into the other, with bending, warping, stretched, but not ripped.

Structural Stability. A phase portrait is structurally stable if its topology cannot be changed by an arbitrary small perturbation to the vector field.

6.4 Rabbits versus Sheep

Here and in ~~future se~~ coming few sections, we are considering examples of phase-portrait analysis.

Lotka-Volterra model of competition and predator-prey. Here, the model of competition will be considered.

Rabbits and sheep compete for food supply (grass), and the grass is limited. This model of competition ignores other factor (predators, seasons, sickness and other food supplies).

Two main effects:

1. (i) Each species would grow to is carrying capacity in the abscence of the other: Logistic model.
   (ii) Rabbits have the famous ability to grow very fast, so the intrinsic growth rate will be considered somehow large.
2. When rabbits and sheep meet, they are in trouble.
   (i) Sometimes the rabbits eat, but the sheep put them away to eat that grass.
   We assume these encounters are proportional to the other species population size. (If there are ~~double~~ as twice as many sheeps, the rabbits will find double sheeps).

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2.(ii). We assume the encounters reduce the growth rate for each species,

2(iii) but the effect is more severe on rabbits.

The model is as follows:

$$\begin{aligned}\dot{x} &= x(3 - x - 2y)\\ \dot{y} &= y(2 - x - y)\end{aligned}\Bigg\}\quad \begin{aligned}&x - \text{Population of rabbits}\\ &y - \text{Population of sheep.}\end{aligned}$$

Note the system can be written as:

1.(ii)   2(ii): Negative sign.

$$\dot{x} = \underbrace{x(3 - x)}_{1.(i)} - \underbrace{2yx}_{2(i)}$$

2(iii).

$$\dot{y} = \underbrace{y(2 - y)}_{1.(i)} - \underbrace{xy}_{2(i)}$$

Here, $x(t) \geq 0$, $y(t) \geq 0$.

<u>Fixed pts.</u> $\begin{aligned}\dot{x} &= 0\\ \dot{y} &= 0\end{aligned}\Big\} \Rightarrow \begin{aligned}x(3 - x - 2y) &= 0\\ y(2 - x - y) &= 0\end{aligned}$

Then, (a) $x = 0$, $y = 0$.

b) $x = 0$, $\underset{y \neq 0}{\Rightarrow}$ $2 - y = 0$, $y = 2$

c) $y = 0$ $\underset{x \neq 0}{\Rightarrow}$ $3 - x = 0$, $x = 3$

d) $x \neq 0$, $y \neq 0 \Rightarrow \begin{aligned}x + 2y &= 3\\ x + y &= 2\end{aligned} \Rightarrow \begin{aligned}x &= 1\\ y &= 1\end{aligned}$

= 159 =

<u>Fixed points:</u>

(0,0)
(0,2)
(3,0)
(1,1).

The Jacobian is computed accordingly to

$$f(x,y) = 3x - x^2 - 2xy$$
$$g(x,y) = 2y - xy - y^2$$

$$J(x,y) = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}$$

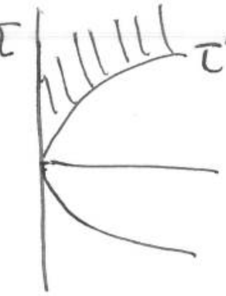
<u>Linear analysis about fixed pts.</u>

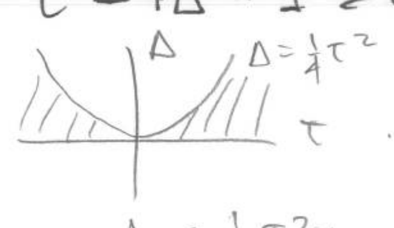
$\boxed{(0,0)}$ $\quad J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. $\quad \lambda_1 = 3,\ \lambda_2 = 2$, $\quad \Delta = 6,\ \tau = 5$

The origin is an <u>unstable</u> node; since $\lambda_1 = 3,\ \lambda_2 = 2 > 0$.

and $\tau^2 - 4\Delta = 1 > 0$.

$\tau^2 - 4\Delta > 0$ means to be to the left of the parabola.





$\Delta < \frac{1}{4}\tau^2$.

Since $\lambda_1 = 3 > \lambda_2 = 2$, then, trajectories leave the origin tangentially to the eivector

$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longleftrightarrow \lambda_2 = 2$, since this is the smallest eival (in absolute value) $|\lambda_2| < |\lambda_3|$

$\boxed{(0,2)}$ $J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$. $\lambda_1 = -1$, $\lambda_2 = -2$

$\Delta = 2 > 0$, $\tau = -3 < 0$. This means stability.

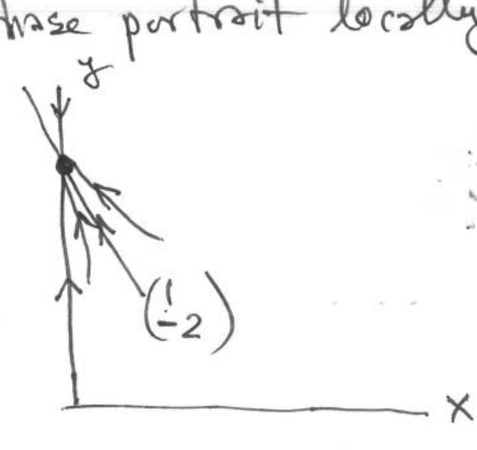
$\tau^2 - 4\Delta = 1 > 0$, this is to the left of the parabola, then this is a <u>stable node</u>.

Similarly, $\lambda_1 = -1$, $\lambda_2 = -2$ implies a <u>stable</u> <u>node</u>.

<u>Eigenvectors</u> $\lambda_1 = -1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = -2 \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The phase portrait locally about $(0,2)$ looks like:



Trajectories approach faster to $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, since $|\lambda_1| < |\lambda_2|$.

$\boxed{(3,0)}$ $J(0,3) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$: $\left.\begin{matrix} \tau = -4 \\ \Delta = 3 \end{matrix}\right\}$ Stable.

$\tau^2 - 4\Delta = 4 > 0$: <u>Node</u>

$\underbrace{\lambda = -1, -3}$ This also implies it is a stable node.

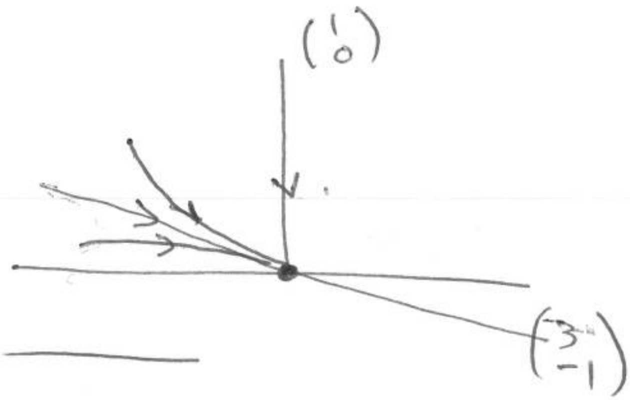
This is a <u>stable node</u>.

Eigenvektors:

$$\lambda_1 = -1 \Rightarrow \bar{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

$$\lambda_2 = -3 \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Sketch near fixed point $(3,0)$



$\boxed{(1,1)}$

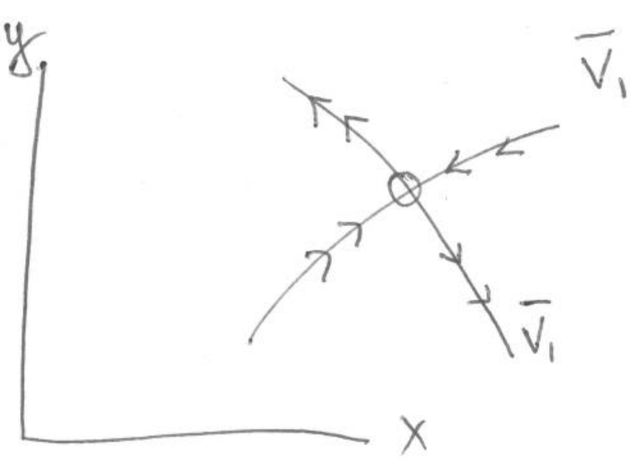
$$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}, \quad \tau = -2$$

$\Delta = -1 \Rightarrow$ saddle point.

Eigenvalues $\lambda_{1,2} = -1 \pm \sqrt{2}$.

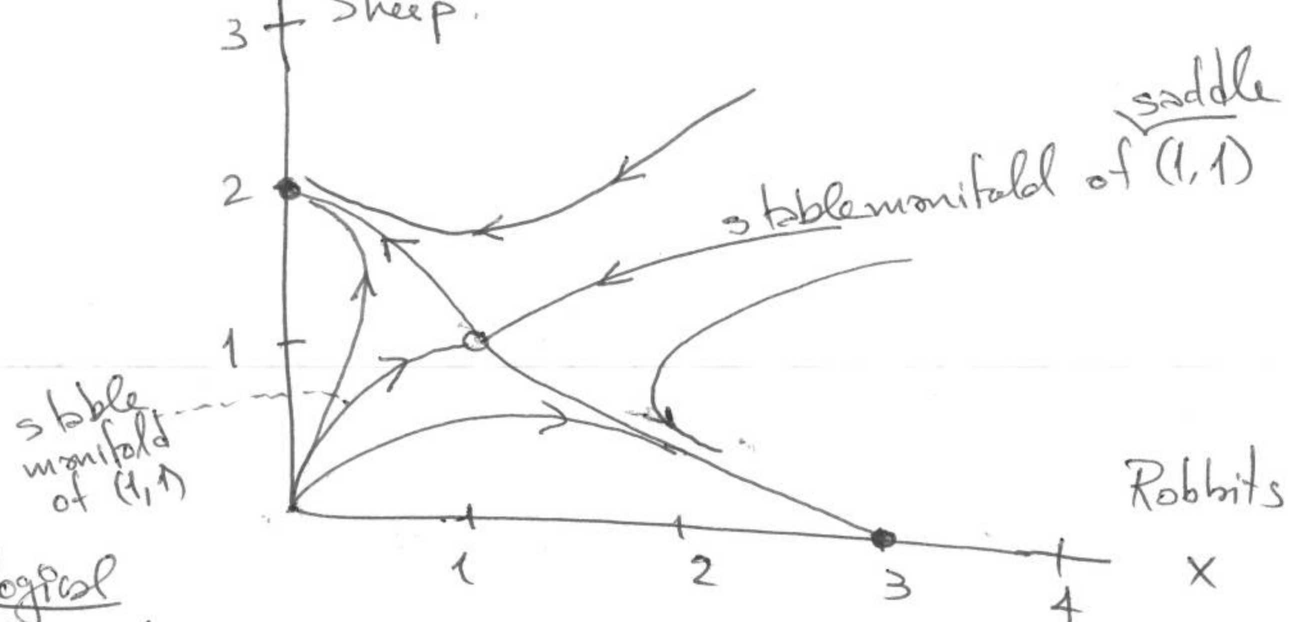
<u>Eigenvectors</u> $\lambda_1 = -1+\sqrt{2} \Rightarrow \bar{v}_1 = \begin{pmatrix} 2 \\ -\sqrt{2} \end{pmatrix}$

$$\lambda_2 = -1-\sqrt{2} \Rightarrow \bar{v}_2 = \begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix}$$



Combining all the figures, we get a sense of the phase portrait.

Note that the x- and y-axis are invariant lines, since $\dot{y}=0$ and $\dot{x}=0$ there.



Biological Interpretation: One species takes the other one to extinction (in most of the cases).

* Starting above the stable manifold, Sheep survive and rabbits die
* Starting under the stable manifold, Rabbits survive and sheep die.
* Principle of competition exclusion (formulated by biologists): Two species competing by the same limited resources typically cannot coexist.

A more general mathematical concept.

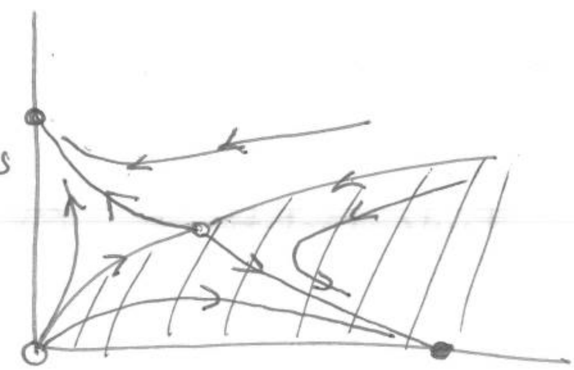
* A basin of attraction.

Given a fixed point, x_f, a basin of attraction is the set of all initial conditions $\vec{x}(0)$, such that

$$\vec{x}(t) \xrightarrow[t \to \infty]{} x_f,$$

for those initial conditions.

* Example. In Example considered in the current section, we called the basin of attraction of node (3,0) all those points below the stable manifold of the saddle:



Since the stable manifold separates the basins for the two nodes, it is called boundary of the basin.

They are also called separatrices.

Basins and boundaries are important since they divide the phase plane into regions of different long-term behaviour.

$\boxed{6.5}$ Conservative Systems. 03310320IO

Newton's law: $m\ddot{\vec{x}} = F(\bar{x})$, where $\bar{x} \in \mathbb{R}^3$,

and $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

m is the mass of the particle under consideration.

Consider the particle moving along the x-direction. Then,

$$m\ddot{x} = F(x).$$

Assume F only depends on x (but not on t or $\dot{x}$), i.e., there is no damping or friction, and there is no driving forces (depending on t).

Under these assumptions, the system is conservative, i.e., there is a quantity which is conserved. In this particular example, the energy is conserved.

Define the potential energy $V(x)$ such that

$$F(x) \equiv -\frac{d}{dx}V$$

Hence, Newton's law in 1-dim becomes:

$$m\ddot{x} + \frac{dV}{dx} = 0$$

i.e, multiply by $\dot{x}$:

$$m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V\right) = 0$$

Then:
$$\frac{1}{2}m\dot{x}^2 + V(x) = \text{const} \equiv E,$$
and we define this constant to be the energy of the system.

The energy is a conserved quantity, a constant of motion or a first integral

Systems with (at least) a conserved quantity are called conservative systems.

More generally:

Let $\dot{\bar{x}} = f(\bar{x})$ be a system. A conserved quantity is a realvalued function $E(\bar{x})$, that is constant along trajectories

i.e.:
$$\frac{dE}{dt} = 0.$$
It is also required that $E(\bar{x})$ is not constant on any open set.

(Otherwise, cases like $E(\bar{x}) \equiv 0$ will qualify for constants of motion and hence, every system would be conservative)

Example 6.5.1. Show that conservative systems cannot have attracting fixed points.

Soln: Let x_f be a fixed pt. All trajectories approaching x_f have the same energy E, since they are attracted to x_f.

Then, any point in the basin of x_f should ~~have~~ have the same energy E.

I.e., all points in the basin have energy E, and $E = E(\bar{x})$ on the basin, which is an open set. This contradicts the requirement that E shouldn't be a constant in open sets.

QED

Example $\boxed{6.5.2}$ Consider a particle of mass $m = 1$, moving under the influence of a potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. Find and classify all its fixed pts, plot the phase portrait, and physically interpret the results.

The force is given by: $F = -\frac{dV}{dx} = x - x^3$.

Hence $\ddot{x} = x - x^3$, and the system reads:

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

Equilibrium soln's: $\begin{matrix} \dot{x} = 0 \\ \dot{y} = 0 \end{matrix}$ simultaneously $\Rightarrow$ $\begin{matrix} y = 0 \\ x(1-x^2) = 0 \end{matrix}$

Hence: $(x_f, y_f) = (0,0), (1,0), (-1,0)$.

The Jacobian is $J(x,y) = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$.

* At $(x_f, y_f) = (0,0)$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\Delta = -1$, $\tau = 0$, then, it is a saddle pt.

* At $(x_f, y_f) = (1,0), (-1,0)$.

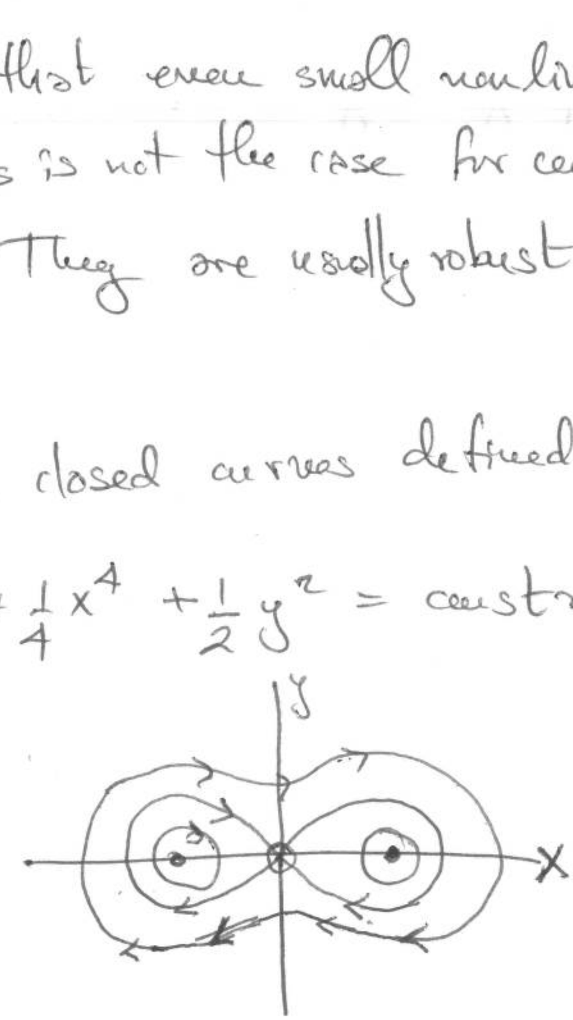
$$J(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

$\Delta = 2$, $\tau = 0$. Then, they are centers, predicted via linear theory.

We learned previously that even small nonlinearity can destroy centers. This is not the case for centers in conservative systems. They are usually robust because the conservation of energy.

The trajectories are closed curves defined by:

$$E = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2 = \text{constant},$$



The system has:

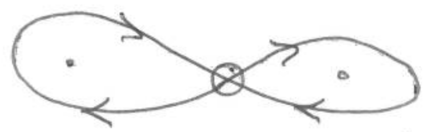
* a saddle at $(0,0)$
* centers at $(1,0)$ and $(-1,0)$

* Each neutrally stable centers are surrounded by a family of small orbits, which are closed orbits
* There are also large orbits enclosing all three fixed point
* The solutions are periodic; except:

(i) fixed points

(ii) "homoclinic orbits"

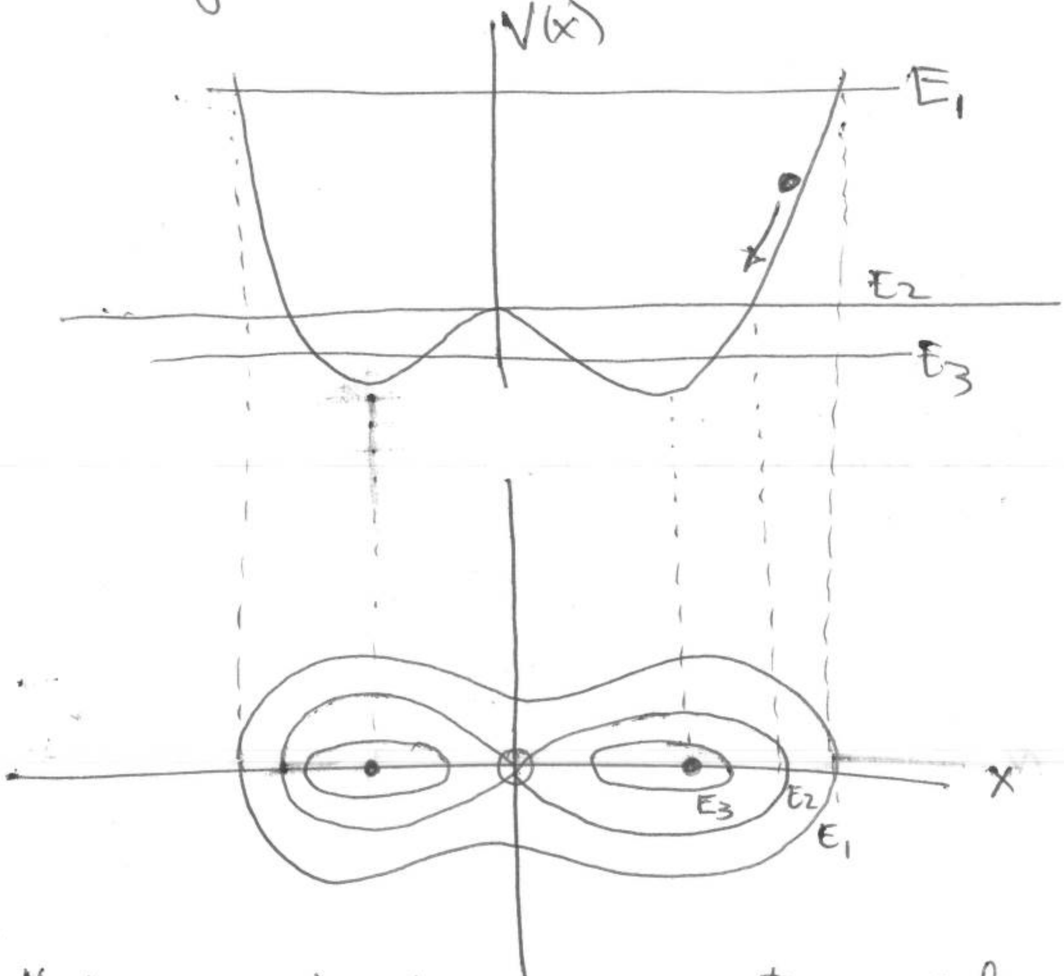
* The homoclinic orbits are those trajectories that (apparently) start and end at the origin.



* "Apparently", in the sense that they never touch the origin, but instead $\vec{x}(t) \xrightarrow[t\to\pm\infty]{} 0$. Otherwise, uniqueness would be violated.
* Trajectories "starting" and "ending" at the same point are called homoclinic orbits.
* Homoclinic orbits are common in conservative systems; rare otherwise.

* ~~Homoclinic orbits~~ do not correspond to periodic orbits, because the trajectory takes forever to reach (x_f, y_f).

* Coming back to the ~~trajectory~~ potential function:



High energy trajectories E_1: Big periodic orbits

Critical energy E_2: Saddles.

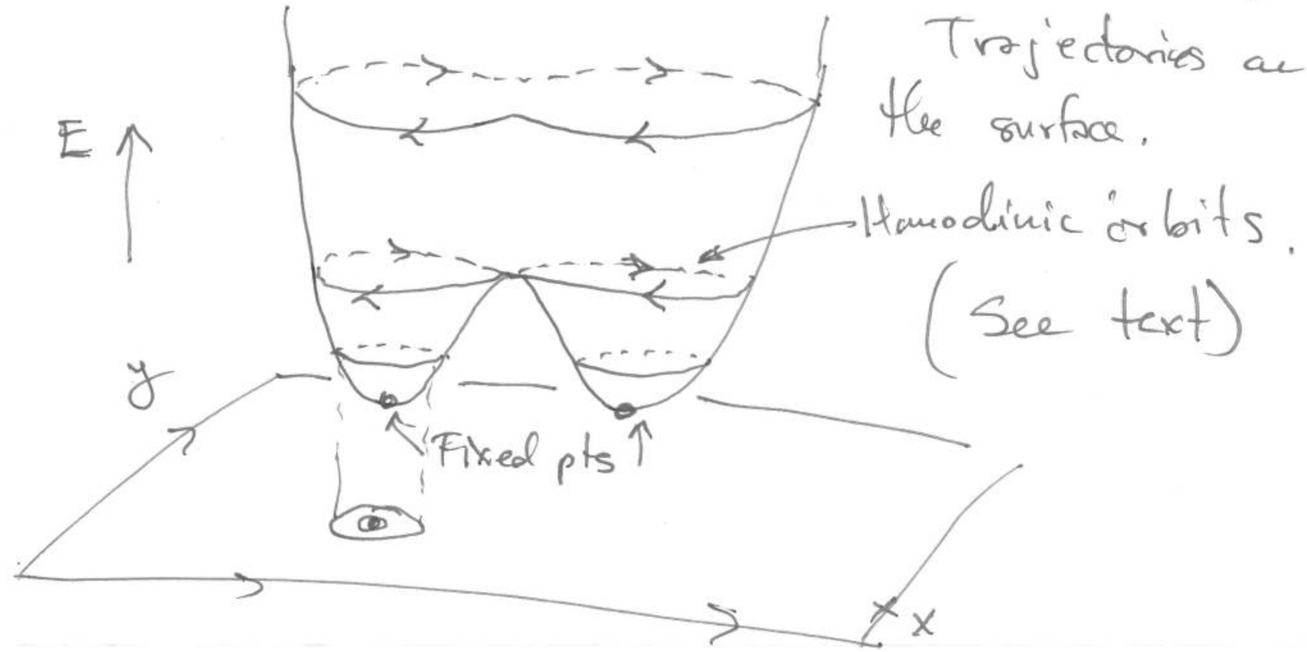
Top of saddle, unstable saddle.

Low energy E_3: Small periodic orbits.

Lowest energy: Fixed points: centers

(See text).

Example $\boxed{6.5.3.}$ Sketch of the energy function: $E = E(x, y)$ 03103 2010.



Nonlinear Centers.

* Centers are very fragile in general. But Centers are, in general, robust when the system is conservative.
  * Centers occur at local minima of the "energy function" $E = E(x,y)$, where the equilibria are neutrally stable and small oscillations may occur, at the bottom of the potential.

Theorem 6.5.1. Nonlinear Centers for Conservative Systems.
Let $\dot{x} = f(\bar{x})$ be a system, with $\bar{x} \in \mathbb{R}^2$, and $f \in C^1(\mathbb{R}^2)$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$.
Assume $E = E(\bar{x})$ is a conserved quantity for the system, and that x_f is an isolated fixed point.
Then, if x_f is a local minimum of E, then trajectories

close enough to x_f are closed orbits

Remarks 1. The theorem is valid for maxima of E also.
2. The requirement that x_f is isolated is required.

6.6 Reversible systems.

Movie of an undamped pendulum looks the same if we watch the movie running forward or reversed.

Newton's law has the time-reversal symmetry:

$$m\ddot{x} = F(x)$$

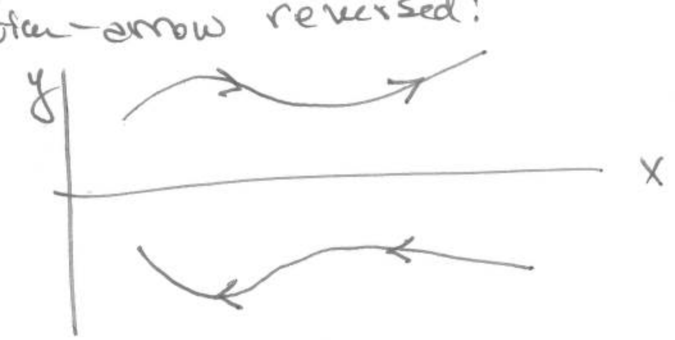
Substitute $t \longrightarrow -t$, and we get the same equation.

Written as a system:

$$\dot{x} = y$$
$$\dot{y} = \frac{1}{m}F(x).$$

We require here the symmetry $\begin{matrix} t \longrightarrow -t \\ y \longrightarrow -y \end{matrix}$. Hence, if $(x(t), y(t))$ is a solution, $(x(-t), -y(-t))$ is also a solution.

There is a symmetry in the phase plane about the x-axis, with the direction-arrow reversed:



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Def: A second-order reversible system is of the form,

$$\dot{x} = f(x,y) \qquad (*)$$
$$\dot{y} = g(x,y)$$

such that $f(x,y)$ is odd in y $\left(f(x,-y) = -f(x,y) \right)$

and $g(x,y)$ is even in x $\left(g(x,-y) = g(x,y) \right)$.

such that the system $(*)$ above is invariant under the transformation: $t \longrightarrow -t$ and $y \longmapsto -y$ (simultaneously)

Theorem: 6.6.1. Nonlinear Centers for reversible systems.

Let $x_f = 0$ be a fixed point of the C^1 system: $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$.

Assume x_f is a center by linear analysis.

Also assume the system is reversible.

Then: sufficiently close to the origin, all trajectories are closed curves.

(Check the text for ideas of the proof)

Example $\boxed{6.6.1}$:

$$\dot{x} = y - y^3$$
$$\dot{y} = -x - y^2.$$

The system has a (nonlinear) center at the origin, and plot the phase portrait.

Soln: The Jacobian is: $J(x,y) = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}$

The origin is a fixed point. Then: $J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Then $\tau = 0$, $\Delta = 1$, and the origin is a center, via linear analysis

Also, the system is reversible: Consider $t \longrightarrow -t$, and $y \longrightarrow -t$

the system stays the same. Then, by Thm 6.6.1., the origin is also a center for the nonlinear system.

The other fixed pts: $y(1-y^2) = 0$

$$-x - y^2 = 0.$$

i) $y = 0 \Rightarrow x = 0$: Origin. (already known).

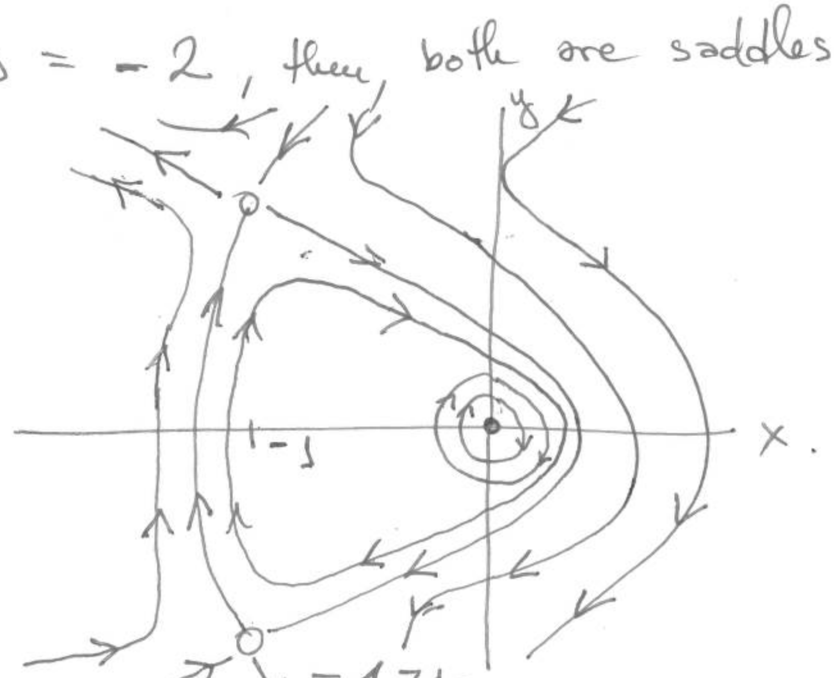
ii) $y \neq 0 \Rightarrow 1 - y^2 = 0 \Rightarrow y = \pm 1 \Rightarrow x = -1.$

$$\Rightarrow (x_1, y_1) = (-1, \pm 1)$$

Evaluate

$$J(x_1, y_1) = \begin{pmatrix} 0 & -2 \\ -1 & \mp 2 \end{pmatrix}$$

$\Delta = -2$, then, both are saddles.



At $y = 0$:

$\dot{x} = 0$

$\dot{y} = -y^2 \leq 0$

* Observe there are two trajectories that joint both saddles. They are called saddle connections or heteroclinic orbits, since the join two different fixed points (as opposed to the homoclinic orbits).

Example $\boxed{6.6.2}$: Show that the system has a homoclinic orbit in the half plane $x \geq 0$:

$$\dot{x} = y$$
$$\dot{y} = x - x^2.$$

Fixed pts. $y = 0 \quad \Rightarrow \quad (x_0, y_0) = (0,0)$

$x(1-x) = 0 \qquad (x_1, y_0) = (1, 0).$

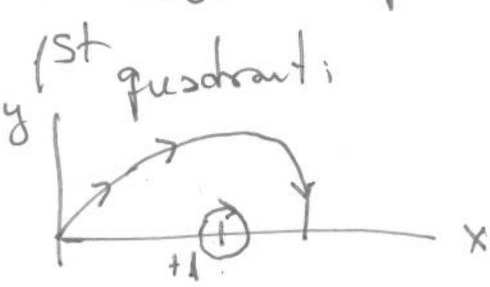
The Jacobian is:
$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$$

Then $J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\Delta = -1 \Rightarrow$ linear saddle fixed point.

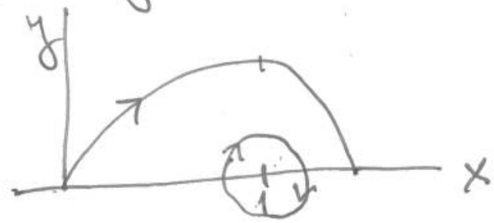
$J(1,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $\left.\begin{matrix} \Delta = +1 \\ \tau = 0 \end{matrix}\right\} \Rightarrow$ linear center..

The system is reversible. Then, the linear center is also a center for the nonlinear system.

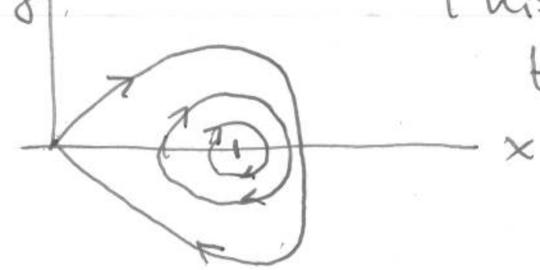
Near the origin, $\dot{x} > 0$, $\dot{y} > 0$, for $(x,y) \in 1^{st}$ quadrant. Then, the unstable manifold is on the 1^{st} quadrant:



As y increases, there is a moment when $\dot{y} = 0$, and this is when $x = 1$, then y decreases $(\dot{y} < 0)$ after $x = 1$, until the trajectory hits the x-axis.



By reversibility there is a reflected trajectory about the x-axis, with arrows reversed:



This is a homoclinic trajectory.

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* More general definition of reversibility.

Let $R: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, such that $R^2(\bar{x}) = \bar{x}$, with $\bar{x} \in \mathbb{R}^n$. (an example of R is a reflection). The system:

$$\dot{x} = f(x)$$

is reversible if it is invariant under $\begin{array}{l} t \longrightarrow -t \\ x \longrightarrow R(x). \end{array}$