# 8 Bifurcations Revisited

From 1-dim to 2-dim systems, fixed pts can be created, destroyed, destabilized as a parameter is detuned.

We begin to describe the way in which oscillations can be turned on or off.

In this context,
what do exactly we mean by a bifurcation?

If the phase portrait changes its topological structure, as a parameter is detuned, we say tha a bifurcation has occurred.

Example: Changes in the number of, or stability of, fixed points, closed orbits, saddle connections ... as the parameter is varied.

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## 8.1 Saddle-Node, Transcritical, and Pitchfork bifurcati

Bifurcations in chapter 3 have its analogous in 2-dim, and in n-dimensions:

All the action is confined to a 1-dim subspace along which bifurcation occurs, while in the extra dimensions flow is either simple attraction or repulsion from the corresponding subspace.

## Saddle-Node bifurcation.

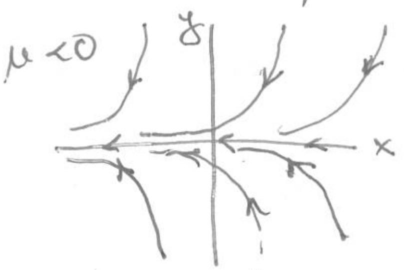
It is the typical mechanism for creation or anhilation of fixed points. The standard model is:

$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$

In the y-direction we have typical attraction, exponential decay to the x-axis.

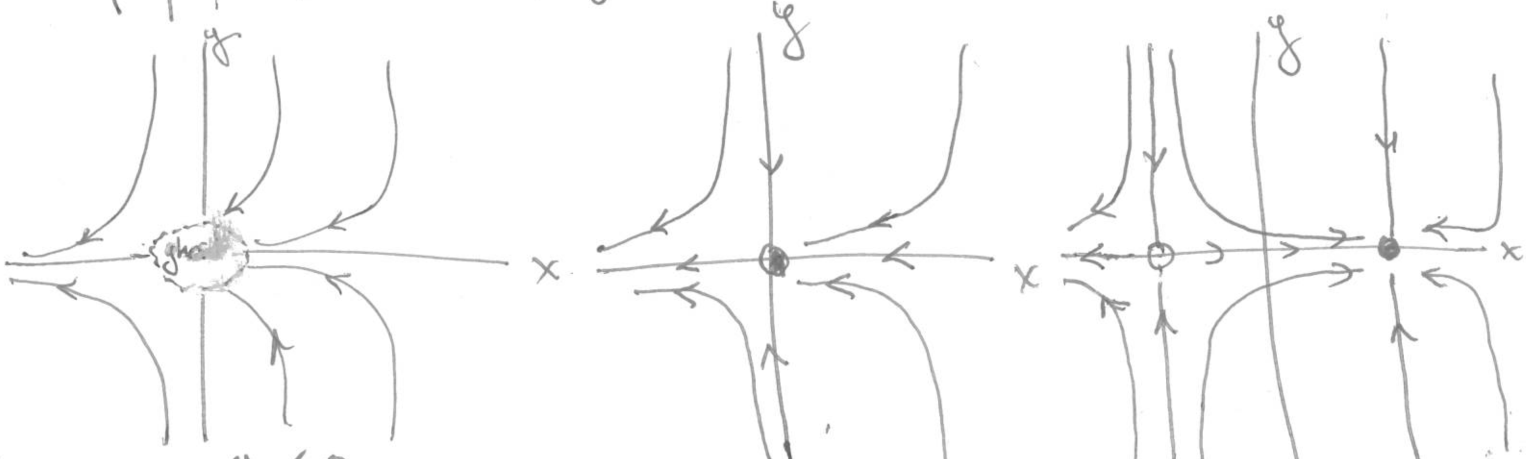
In the x-axis we have the process of bifurcation, as $\mu$ is defined.

For $\mu < 0$, $\dot{x} < 0$ always. When $\mu$ is close to 0, but still negative, there is a "ghost" or "bottle neck" that damps the flux to go through the origin. As it was shown in section 4.3, the time to go through this bottleneck goes as:




$$\frac{1}{\sqrt{|\mu - \mu_c|}},$$ where $\mu_c$ is the value of the parameter for the bifurcation to occur.

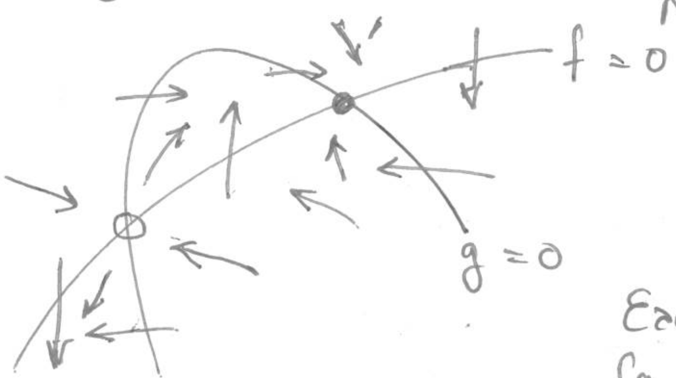
Figure 8.1.1.




$\mu < 0$. $\mu \sim 0^-$ | $\mu = 0$ | Saddle-node $\mu > 0$

Typical picture for a saddle-node scenario

Null-clines for the system




$$\dot{x} = f(x,y)$$

$$\dot{y} = g(x,y)$$

Each intersection corresponds to a fixed point.

We assume that at some value of the parameter, $\mu_c$, the curves $f=0$ and $g=0$ become tangent, and pull away from each other, then the fixed points (one being a saddle and the other a node) collide to desappear in a "blue-sky" collision.

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<u>Example</u> The biological argument can be found in the text.

$$\dot{x} = -ax + y$$

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

where $a>0$, $b>0$

Assume $b$ - fixed.

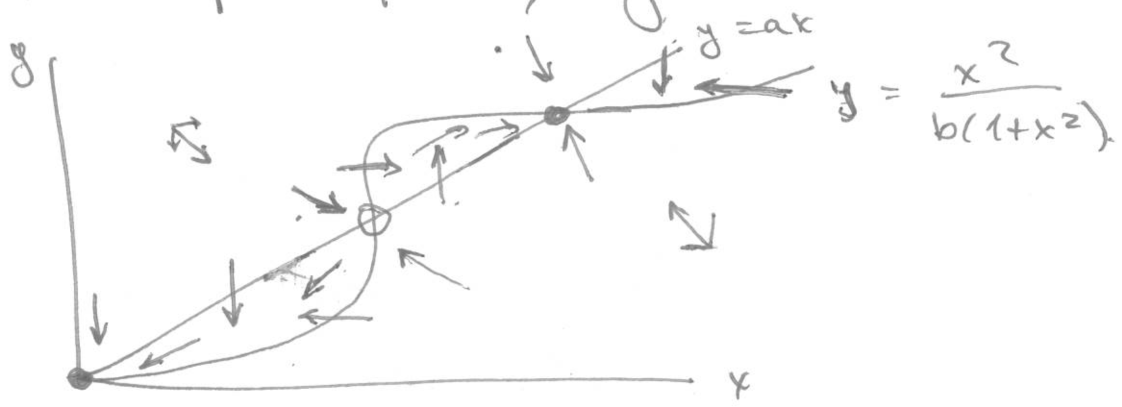
- Show that the system has three fixed points for $a<a_c$.
- Determine $a_c$.
- Show these three points collide at $a=a_c$ in a saddle-node bifurcation.
- Sketch the phase-portrait.
- Interpret biologically.

Solution Null-clines.

$$y = ax \quad \Leftrightarrow \dot{x} = 0 \quad \text{Vertical field}$$

$$y = \frac{x^2}{b(1+x^2)} \quad \Leftrightarrow \dot{y} = 0 \quad \text{Horizontal field}$$

In the phase-plane, they look like




* Notice that above the straight line: $y > ax \Rightarrow y - ax > 0$
  $\Rightarrow \dot{x} > 0 \Rightarrow$ motion to the right.

* Below the line: $y < ax \Rightarrow y - ax < 0 \Rightarrow \dot{x} < 0$.
  $\Rightarrow$ motion to the left

* Above the curve $g(x,y) = 0 (= \dot{y}) \Rightarrow y > \frac{x^2}{b(1+x^2)} \Rightarrow 0 > \frac{x^2}{(1+x^2)} - by$
  $\Rightarrow \dot{y}$ 0, motion goes down

* and below the curve $g(x,y) = 0 \Rightarrow y < \frac{x^2}{b(1+x^2)} \Rightarrow 0 < \frac{x^2}{\cancel{b}(1+x^2)} - by$
  $\dot{y} > 0$, motion goes up.

We observe we have:

node – saddle – node.

for values of $a$ small enough.

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For large values of $a$, the line has a large slope, having just one intersection, the origin.

To find the intersections i.e. the fixed pts: and the value of $a_c$ when the bifurcation occurs, let's solve: $f(x,y)=0$
$g(x,y)=0$.

$$ax = \frac{x^2}{b(1+x^2)}.$$

Then, the origin: $x=0$, $y=0$ is a solution.

If $x \neq 0$ (then $y \neq 0$) and so:

$$a = \frac{x}{b(1+x^2)}$$

i.e. $$ab(1+x^2) = x$$

$$\Rightarrow \quad abx^2 - x + ab = 0$$

then, the fixed pts are given by (the x-components):

$$x_f = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}$$

Notice there are <u>two</u> solutions for $1-4a^2b^2 > 0$
i.e. for $2ab < 1$

Then, the fixed points collide when: $2ab = 1$, i.e.

$$a_c = \frac{1}{2b} \quad \text{(remember } b \text{ is fixed)}$$

Notice that the bifurcation occurs when $a = a_c = \frac{1}{2b}$ and for

$$x_f^{a_c} = \frac{1 \pm \sqrt{1 - 4\frac{1}{(2b)^2}b^2}}{2\left(\frac{1}{2b}\right)b} = \frac{1 \pm \sqrt{1-1}}{1}$$

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I.e., the bifurcation occurs at

$$a = a_c \Rightarrow \begin{array}{l} x_f^{a_c} = 1. \\ y_f^{a_c} = a_c = \dfrac{1}{2b} \end{array}$$

We can check what type are the fixed point are for $a < a_c$ by means of linear analysis.

$\boxed{\text{At the origin: } (0,0)}$

$$\begin{array}{l} \dot{x} = -ax + y \\ \dot{y} = \dfrac{x^2}{1+x^2} - by \end{array} \Rightarrow J(x,y) = \begin{pmatrix} -a & 1 \\ \dfrac{2x}{(1+x^2)^2} & -b \end{pmatrix}$$

$$g_x = \frac{(1+x^2)2x - x^2 2x}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

Then:
$$J(0,0) = \begin{pmatrix} -a & 1 \\ 0 & -b \end{pmatrix}$$

* $\tau = -(a+b) < 0. \Rightarrow$ stable sink or saddle.

* $\Delta = ab > 0 \Rightarrow$ stable sink $\Rightarrow \left\{ \begin{array}{l} \text{spiral} \\ \text{or} \\ \text{node.} \end{array} \right.$

* $$\begin{aligned} \tau^2 - 4\Delta &= (a+b)^2 - 4ab \\ &= a^2 + 2ab + b^2 - 4ab = a^2 - 2ab + b^2 \\ &= (a-b)^2 > 0. \quad (\text{if } a \neq b) \end{aligned}$$

left to the parabola: $4\Delta < \tau^2$
the Origin is a stable node.

$\boxed{\text{At the other fixed points:}}$

* $\tau = -(a+b) \Rightarrow$ still a saddle or $\left\{\begin{array}{l}\text{stable}\\\text{sinks.}\end{array}\right.$

* Here $\Delta = ab - \dfrac{2x_f}{(1+x_f^2)^2} \underset{\uparrow}{=} ab - \dfrac{2ab(1+x_f^2)}{(1+x_f^2)^2} =$

i.e. $\quad ab(1+x_f^2) = x_f$

$$\Delta = ab\left[1 - \frac{2}{(1+x_f^2)}\right] = ab\left(\frac{x_f^2-1}{(1+x_f^2)}\right).$$

* Now, for

$$x_f = \frac{1 - \sqrt{1-4a^2b^2}}{2ab} < 1.$$

notice that

$$1 - \sqrt{1-4a^2b^2} < 2ab \Rightarrow \underbrace{1-2ab}_{\substack{\text{positive:}\\\text{since } 2ab<1}} < \sqrt{1-4a^2b^2} \Rightarrow$$

$$\Rightarrow (1-2ab)^2 < 1-4a^2b^2 \Rightarrow 1-4ab+4a^2b^2 < 1-4a^2b^2$$

$$\Rightarrow 8a^2b^2 < 4ab \Rightarrow 2ab < 1 \text{ and go backwards}$$

Hence, since $x_f < 1$: $\Delta = ab\left(\dfrac{x_f^2-1}{(1+x_f^2)}\right) < 0$,

thus, this fixed point is <u>a saddle</u>.

* For: $x_f = \dfrac{1+\sqrt{1-4a^2b^2}}{2ab} > 1$

This is true because:

$$1+\sqrt{1-4a^2b^2} > 2ab$$
$$\sqrt{1-4a^2b^2} > 2ab-1$$
$$\sqrt{1-4ab^2} \geq 0 > 2ab-1$$

and go backwards.

Hence

$x_f > 1$ implies: $\Delta = ab\left[\dfrac{x_f^2-1}{1+x_f^2}\right] > 0.$

then, this fixed point is a <u>sink</u>.

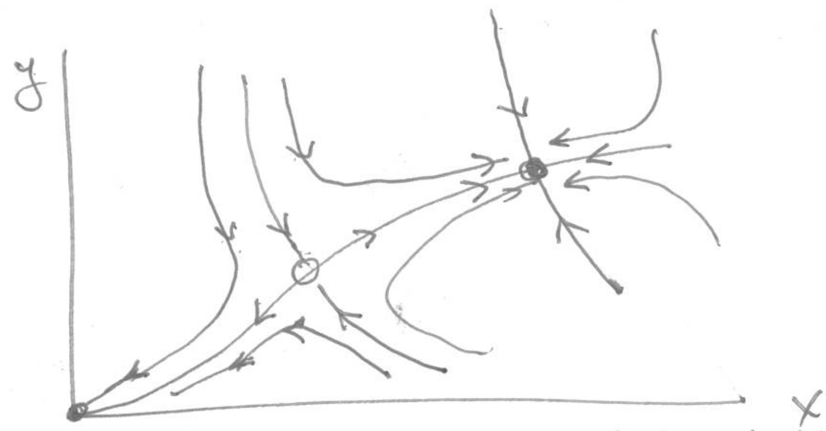
Actually, $\Delta = ab\left(\dfrac{x_f^2-1}{1+x_f^2}\right) < ab$. Then:

$$\begin{aligned}\tau^2 - 4\Delta &= (a+b)^2 - 4\Delta = a \\ &> (a+b)^2 - 4ab = a^2+2ab+b^2-4ab \\ &= a^2-2ab+b^2 = (a-b)^2 \geq 0.\end{aligned}$$

ie: $\tau^2 > 4\Delta$, ie. $4\Delta < \tau^2$, we are to the left of the parabola

and the <u>sink</u> is a <u>stable node</u>.

Looking at the figure with nullclines in it, we observe that the unstable manifold of the saddle lies in between the nullclines. The phase portrait looks like:




For a biological interpretation, look at the text.

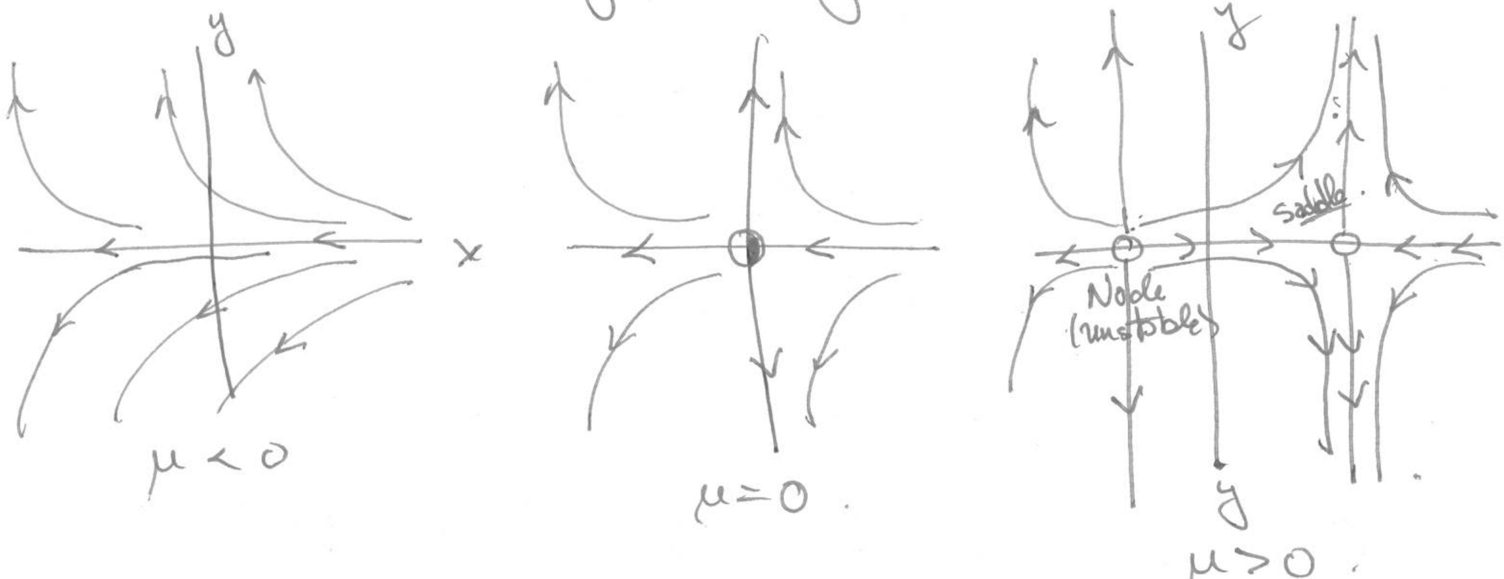
We observe that the picture looks pretty similar to the prototypical saddle-node bifurcation Figure 8.1.1. All trajectories fastly decay onto the unstable manifold of the saddle point..

→ The bifurcation is, basically, 1-dimensional
→ They are building block in higher dimensional systems.
→ This can be justified by the Center Manifold theorem
→ This is the reason why 1-dim bifurcations were studied in deep detail.
→ Details can be found in Wiggins, Introduction to Applied Nonlinear Dynamical System and Chaos (1990); Center Manifold Theory.

Remark: If

$$\dot{x} = \mu - x^2$$
$$\dot{y} = +y$$

The horizontal behavior remains the same, but we now we have a flux that goes away from the horizontal axis.




We still have a saddle-node bifurcation, but now it turns that the node becomes unstable.

## Transcritical and Pitchfork Bifurcations.

The prototypical examples are:

| | | |
|---|---|---|
| $\dot{x} = \mu x - x^2$ | $\dot{x} = \mu x - x^3$ | $\dot{x} = \mu x + x^3$ |
| $\dot{y} = -y$ | $\dot{y} = -y$ | $\dot{y} = -y$ |
| Transcritical bifurcation | Supercritical pitchfork bifurcation | Subcritical pitchfork bifurcation |

Example 8.1.2.

Construct the phase portrait for the supercritical pitch-fork bifurcation system:

$$\dot{x} = \mu x - x^3$$
$$\dot{y} = -y$$

for the several cases of $\mu$-values.

Soln:

For the one-diml case:

$\mu < 0$ $\quad \dot{x} = \mu x - x^3 < 0$ for $x > 0$; $\dot{x} = \mu x - x^3 > 0$ for $x < 0$ } Then $x_f = 0$ is stable and it is the only fixed point.

If $\mu \geq 0$ $\quad \dot{x} = (\mu - x^2)x \Rightarrow$ 3 fixed points: The origin $x = 0$ and $x = \pm\sqrt{\mu}$

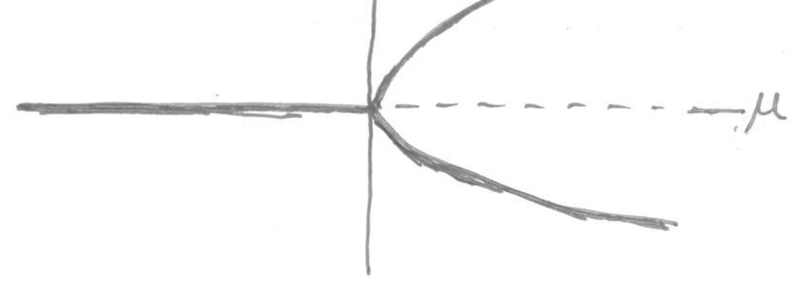
Now, the origin is unstable:

$$f'(x) = \mu - 3x^2$$
$$f'(0) = \mu > 0$$

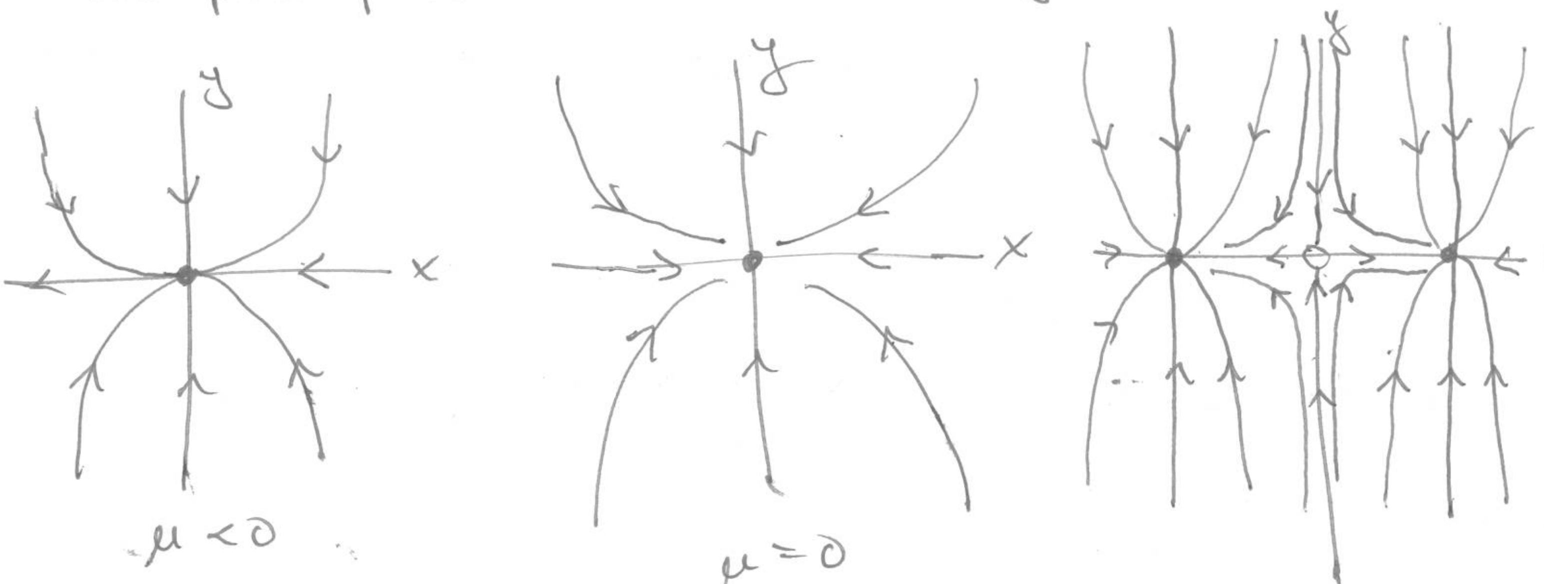
However: $f'(\pm\sqrt{\mu}) = \mu - 3\mu = -2\mu < 0$

Then $x_f = \pm\sqrt{\mu}$ are stable

The bifurcation diagram:



The phase portraits for the 2-dim'l system are;



$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$

Here, the decay in the x-direction is not exponential, but algebraic.

This type of systems occur very often in Physics

**Example:** $\boxed{8.1.3}$ Show that a pitchfork bifurcation occurs at the origin for the system:

$$\dot{x} = \mu x + y + \sin x$$
$$\dot{y} = x - y,$$

and determine the value of the bifurcation parameter, $\mu_c$, at which the bifurcation take place. Plot the phase portrait near the origin for $\mu \gtrsim \mu_c$.

Solution Notice the system is invariant for $\begin{matrix} x \longrightarrow -x \\ y \longrightarrow -y. \end{matrix}$

Then, there is a symmetry w.r.t. the origin.
Then, the origin should be a fixed point.

The Jacobian is:

$$J(x,y) = \begin{pmatrix} \mu + \cos x & 1 \\ 1 & -1 \end{pmatrix}.$$

At the <u>origin</u>:

$$J(0,0) = \begin{pmatrix} \mu + 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Here:

$$\tau = \mu$$

$$\Delta = -(\mu + 2).$$

* The origin is a <u>saddle</u> if: $\Delta < 0$, i.e., $-(\mu+2) < 0$.

  i.e. if:

$$-2 < \mu.$$

* And if:

$$\mu < -2$$

then: $\Delta > 0$ and $\tau = \mu < -2 < 0$,

then the <u>origin</u> is a <u>stable</u> <u>sink</u>. In fact:

$$\tau^2 - 4\Delta = \mu^2 + 4(\mu + 2) = \mu^2 + 4\mu + 8$$

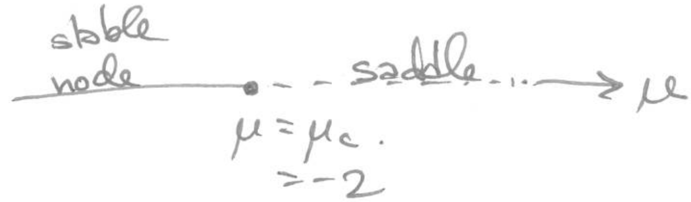
$$= (\mu + 2)^2 + 4 > 0.$$

i.e.:

$\Delta < \dfrac{\tau^2}{4}$, we are to the left of the parabola: <u>Stable node</u>.

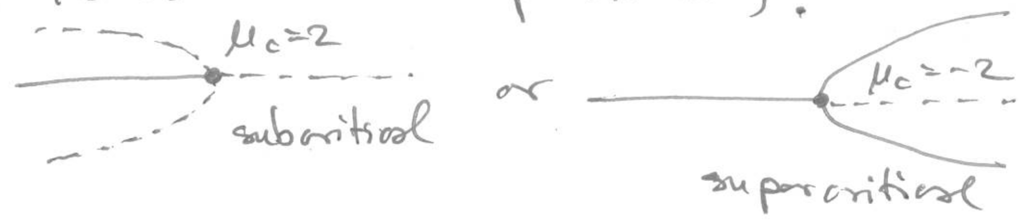
The important part is that there is a bifurcation at $\mu_c = -2$ which goes from a <u>stable node</u> to a <u>saddle</u>. Notice we are passing through $\Delta = 0$, i.e., at least one of the eivalues goes zero.

The change from a stable node, to a saddle, suggest a pitchfork bifurcation.




We now look for symmetric fixed points near the origin, for $\mu \simeq -2$, on either side of $\mu_c=-2$. (we still have to check if it is sub- or supercritical).

Two possible scenarios:




Fixed points: $\dot{x}=0 \qquad \mu x + y + \sin x = 0$

$\dot{y}=0 \qquad x - y = 0$

$$\Rightarrow \begin{cases} y = -\mu x - \sin x \\ y = x \end{cases}$$

$$\Rightarrow \boxed{(\mu+1)x + \sin x = 0.} \quad \cdots (*)$$

A solution is $x_f=0$, $y_f=0$, as we already knew.

Consider now eq (*) for $x \sim x_f = 0$, and expand in Taylor.

$$(\mu+1)x + x - \frac{x^3}{6} = O(x^5)$$

$$(\mu+2)x - \frac{x^3}{6} = O(x^5)$$

Since $x \neq 0$: $\quad (\mu+2) - \frac{x^2}{6} = O(x^4)$

Neglect higher order terms to:

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$$x^2 - 6(\mu+2) = 0$$

$$\Rightarrow \quad \boxed{x_f = \pm\sqrt{6(\mu+2)}}$$

* For $\mu+2<0$, i.e. $\mu<-2$, we only have one fixed pt: $x_f=0$.

* For $\mu+2>0$, ie $-2<\mu$, we have three fixed pts:

$$\left\{\begin{array}{l} x_f = 0 \\ x_f = \pm\sqrt{6(\mu+2)} \end{array}\right\}$$ i.e., we have a supercritical Pitchfork bifurcation

* For $\mu>-2$, we know $x_f=0$ is a saddle..
* Since it is supercritical, with no proof we know the pair of new fixed pts are stable..

The Jacobian is, at the origin:

$$\frac{\partial(f,g)}{\partial(x,y)}(0,0) = \begin{pmatrix} \mu+1 & 1 \\ 1 & -1 \end{pmatrix}$$

with $\Delta = -(\mu+1)-1$

ie $\Delta = -(\mu+2)$

Notice that, at $\mu=\mu_c=-2$:

$$\Delta = 0.$$

i.e. one of the eivals is zero: $\lambda_1 = 0$.

Then, at $\mu=\mu_c=-2$

$$\frac{\partial(f,g)}{\partial(x,y)}(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

The eigenvalues are:

$$\lambda_1 = 0 \longrightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2 \longrightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\mu \gtrsim \mu_c$:

$$J = \begin{pmatrix} \mu+1 & 1 \\ 1 & -1 \end{pmatrix}$$

with the eigenvalues: $\det(J - \lambda I) = 0$ ; $\det \begin{pmatrix} (\mu+1)-\lambda & 1 \\ 1 & -(1+\lambda) \end{pmatrix} = 0$

$$\Rightarrow \quad (\lambda - (\mu+1))(\lambda+1) - 1 = 0$$

$$\lambda^2 + \lambda(1 - (\mu+1)) - (\mu+1) - 1 = 0$$

$$\lambda^2 - \mu\lambda - (\mu+2) = 0.$$

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 + 4(\mu+2)}}{2} \qquad \boxed{\lambda = \frac{\mu \pm \sqrt{(\mu+2)^2 + 4}}{2}}$$

For $\mu = -2$: $\lambda = \frac{-2 \pm \sqrt{0 + 4}}{2} = \frac{-2 \pm 2}{2} = \begin{cases} 0 \\ -2. \end{cases}$ ✓

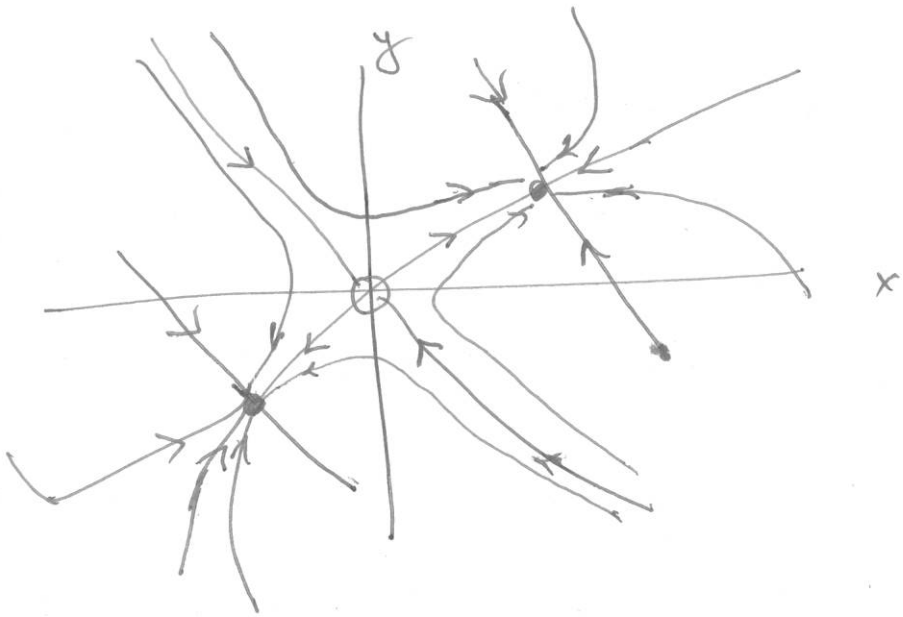
For $\mu \gtrsim -2$, $\lambda_1 = \frac{\mu + \sqrt{(\mu+2)^2 + 4}}{2} \gtrsim 0 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the unstable manifold of the saddle.

We can now plot the phase portrait, being careful that this portrait is valid only for:

$$(x, y) \sim (x_f, y_f) = (0, 0)$$

and

$$\mu \gtrsim \mu_c = -2$$




In all the examples, the bifurcations occur when

$$\Delta = 0,$$

i.e., one at least one of the eival's is <u>zero</u>.

Then: saddle-node, transcritical, pitchfork bifurcation $\Rightarrow$ <u>zero-eigenvalue bifurcations</u>

These are not inclusive (there are more examples) but these are the most common.

Here, we have <u>collision</u> of fixed pts.

The type of bifurcation which follows does not have a 1-dim'l analogous.

This type of bifurcation provides a way to lose stability without the collision of fixed pts.

## $\boxed{8.2}$ Hopf Bifurcations.

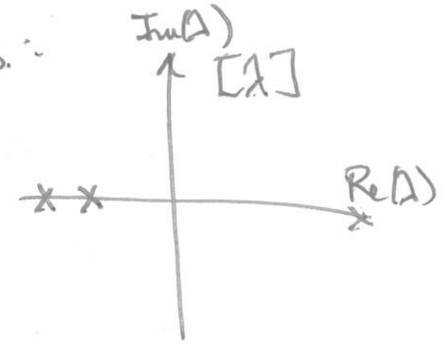
Assume $\vec{x_f}$ is a fixed pt of $\begin{array}{l}\dot{x} = f(x,y,\mu).\\ \dot{y} = g(x,y;\mu).\end{array}$

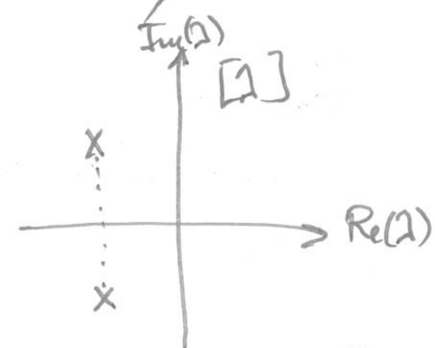
How $\vec{x_f}$ can lose stability as we detune $\mu$?

When linearizing, we get two eivals: $\lambda_1, \lambda_2$

Assume $\vec{x_f}$ is stable then: $\begin{array}{l}Re(\lambda_1) < 0\\ Re(\lambda_2) < 0\end{array}$

Now, $\lambda_1, \lambda_2$ satisfy a quadratic eqn, with real coefficients. So: either they are real, or come in complex conjugates:




To lose stability, we need: (a) one of the real and negative eigenvalues crosses the origing becoming positive, or

(b) both complex conjugates move to the plane Re(λ).

Case (a) was already study in section $\boxed{8.1}$ (and in chapter 3).

Case (b) will be the subject of study for this section.

<u>Supercritical</u> <u>Hopf</u> bi<u>furcation.</u>

Physical system w/equilibrium after oscillations:

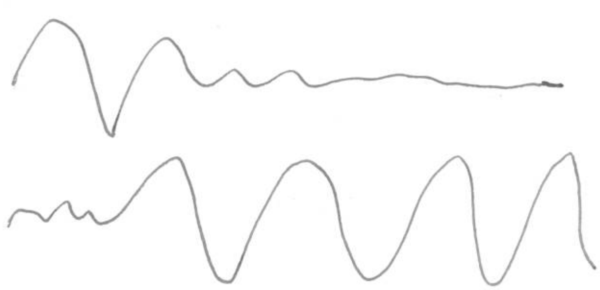
$\Rightarrow$ stable spiral.

The decay rate depend on a parameter $\mu$.

As we detune $\mu$,

the decay gets smaller and smaller, up to a value $\mu_c$, at which the oscillation don't decay, but they growth to a motion of small-amplitude, sinusoidal, becoming a limit cycle;

$\Rightarrow$ this is a <u>supercritical</u>, <u>Hopf</u> bi<u>furcation</u>



$\mu < \mu_c$

$\mu > \mu_c$

In the phase plane:

Hopf bifurcation: Stable spiral $\longrightarrow$ Unstable spiral + small limit cycle

Hopf bifurcations occur if the dimension $n$ of the phase plane is; $n \geq 2$.

Model examples:

$$\dot{r} = \mu r - r^3$$

$$\dot{\theta} = \omega + b r^2.$$

$\mu$ - bifurcation parameter: it controls the stability of the origin $r=0$.

$\omega$ - small oscillations frequency parameter

$b$ - it determines the dependence of the frequency for large amplitude oscillations.

For $\mu < 0$, $\dot{r} < 0$, then $r_f = 0$ is a stable spiral. and the rotation depends on the sign of $\omega$.

For $\mu = 0$, the origin is still a stable spiral, $\dot{r} = -r^3$, but the decay is algebraic. (See Fig. 6.3.2 in text. Remember that linearization predicts a linear center, but not necessarily a center for the nonlinear system).

For $\mu > 0$: $\dot{r} = (\mu - r^2) r$. We have the origin $r_f = 0$, to be an unstable spiral, and a limit cycle $r_f = \sqrt{\mu}$.

We now write the system into cartesians coordinates to study the behavior of the eigenvalues at $\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \vec{0}$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\dot{x} = \dot{r}\cos\theta + r(-\sin\theta)\dot{\theta} = (\mu r - r^3)\cos\theta - r\sin\theta(\omega + br^2)$$

$$= (\mu - r^2) r\cos\theta - r\sin\theta(\omega + br^2) = (\mu - r^2)x - y(\omega + br^2)$$

$$= \mu x - \omega y + \underbrace{(x^2+y^2)(-x - by)}_{\text{cubic terms.}}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = r(\mu - r^2)\sin\theta + r\cos\theta(\omega + br^2)$$

$$= (\mu - r^2)y + x(\omega + br^2) = \mu y + \omega x - r^2(y - bx)$$

$$= \mu y + \omega x \underbrace{- (x^2+y^2)(y - bx)}_{\text{cubic term.}}$$

Linearization about the origin results into (dropping the NL terms).

$$\begin{aligned} \dot{x} &= \mu x - \omega y \\ \dot{y} &= -\omega x + \mu y \end{aligned} \Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix}^{\bullet} = \begin{pmatrix} \mu & -\omega \\ +\omega & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$$

The eigenvalues are given by:

$$\lambda = \mu \pm i\omega$$

As expected, the eigenvalues come in conjugate pairs, and moves from $Re(\lambda) < 0$ to $Re(\lambda) > 0$, as we move $\mu$ from $\mu < 0$ to $\mu > 0$.

Generic properties for Hopf Bifurcations

1. The limit cycle grows as $\sqrt{\mu-\mu_c}$, for $\mu \gtrsim \mu_c$.
for $\mu \lesssim \mu_c$.

2. The frequency of oscillation of the limit cycle is.

$$\omega = \mathrm{Im}(\lambda), \quad \text{at } \mu = \mu_c .$$

Once the limit cycle arises, the frequency correction is:

$$\omega = \mathrm{Im}(\lambda) + O(\mu-\mu_c), \quad \text{for } \mu \gtrsim \mu_c .$$

why?

For $\dot{r} = 0 \Rightarrow r_f = 0$ or $r_f = \sqrt{\mu}$.

Then, at $r_f = \sqrt{\mu} \Rightarrow \dot{\theta} = \omega + b\mu = \omega + b(\mu - \mu_c)$

correction.

The period becomes

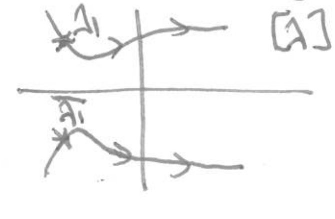
$$T = \frac{2\pi}{\mathrm{Im}\,\lambda} + O(\mu-\mu_c) .$$

---

This idealized prototype of a Hopf bifurcations bring some odds.

(a) The limit cycle is, in general, an ellipse.

(b) The transition from $\mathrm{Re}(\lambda) < 0$ to $\mathrm{Re}(\lambda) > 0$ is not ~~gene~~ generally on a curve (not a straight line)




as we defined $\mu$.

07 05 02 2010.

# Subcritical Hopf bifurcation.

Hopf bifurcation $\left\{ \begin{array}{l} \text{Super critical} \leftarrow \text{Done.} \\ \text{Sub-critical} \end{array} \right.$

Subcritical. Hopf. bifurcation: after the bifurcation occurs, solution jump to distant attractors: another fixed point,
or another limit cycle
or infinity
or a chaotic attractor, (Lorez equations)
(Chapter 9).

— c —

Prototypical example:

$$\dot{r} = \mu r + r^3 - r^5,$$

$$\dot{\theta} = \omega + br^2,$$

In this example, the cubic term $r^3$ is now a destabilizing term.

* For $\mu < 0$: $\quad \dot{r} = r(\mu + r^2 - r^4)$.

Fixed points: $r(\mu + r^2 - r^4) = 0 \Rightarrow r_0 = 0$

and $r^4 - r^2 - \mu = 0$

$\Rightarrow \quad r_{1,2}^2 = \dfrac{1 \pm \sqrt{1+4\mu}}{2}$. If $\mu < 0$: $r_1^2 = \dfrac{1 - \sqrt{1+4\mu}}{2} <$

~~and there is no real values for~~

Actully, for $-\frac{1}{4} < \mu < 0$, $0 < r_1^2 < 1. \Rightarrow 0 < r_2 < 1$

and: $0\ r_2^2 > 1 \Rightarrow r_2 > 1$.

There are two orbits $\begin{array}{l} r = r_2 > 0, \text{ unstable} \\ r = r_1 > 0, \text{ stable} \end{array}$

$f(r) = r(\mu + r^2 - r^4) \quad f'(r) = \mu + 2r^2 - 5r^4$.

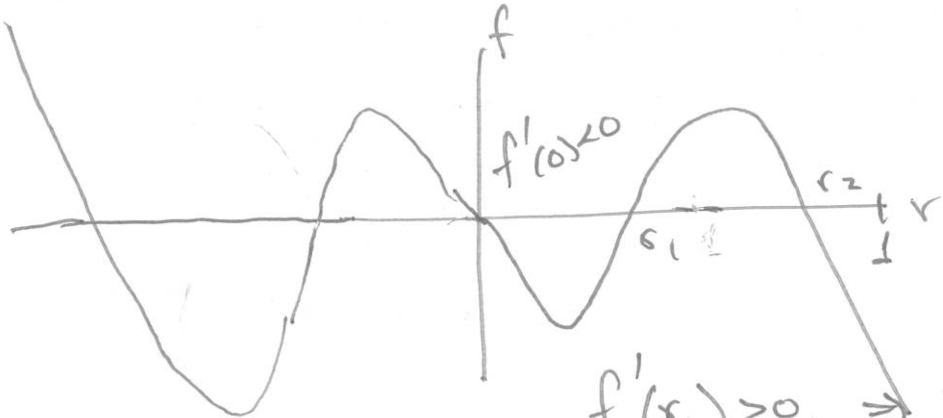
At $r=0$:

$$f'(0) = \mu < 0 \Rightarrow r_0 = 0 \text{ is stable.}$$

Now:

$$f(r) = r(\mu + r^2 - r^4), \quad \mu < 0.$$

$f(r)$ looks like:




$f'(r_1) > 0 \Rightarrow r = r_1$ is unstable

$f'(r_2) < 0 \Rightarrow r = r_2$ is stable.

* At $\mu = 0$ $\dot{x} = f(r) = r^3(1 - r^2)$.

$f(r) = 0$ if $r_0 = 0$ or $r_2 = 1$.

Only two fixed pts.

$f'(r) = 3r^2 - 5r^2$

$f'(0) = 0$ inconclusive.

Similarly: $f'(r_2) = 3 - 5 = -2 < 0$ stable.

The origin becomes then unstable.

(Notice that, for $\mu < 0$: $r_2 \longrightarrow 1$ as $\mu \longmapsto 0^-$

$r_1 \longrightarrow 0$.)

* For $\mu > 0$: Here, $r_1^2 < 0$, and disappear, while $r_2$ grows with $\mu$:

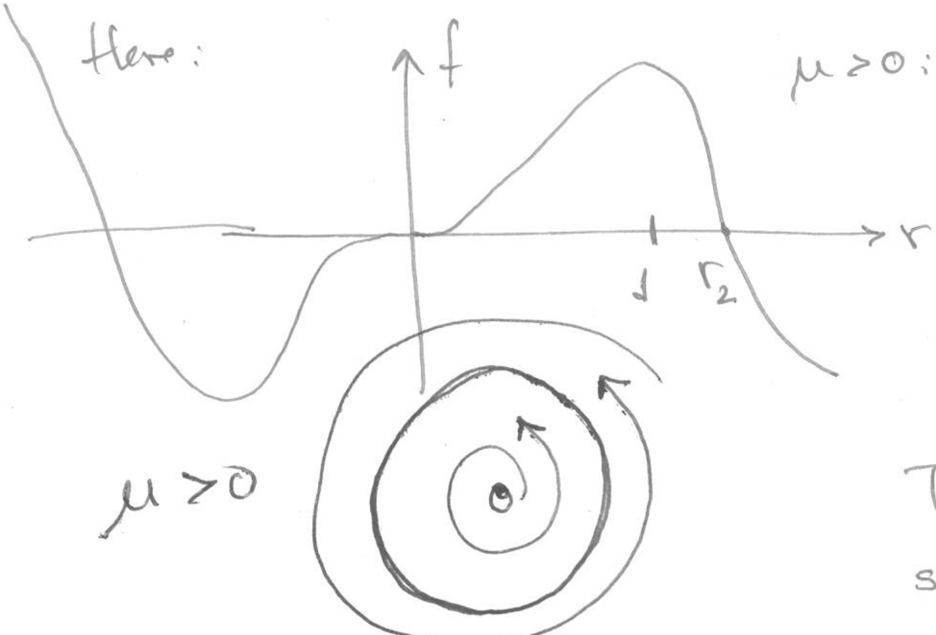
$$r_1^2 = \frac{1 - \sqrt{1 + 4\mu}}{2}$$

$$r_2^2 = \frac{1 + \sqrt{1 + 4\mu}}{2}$$

=222

Here:

$\mu > 0$:




Notice that $r_2 > 1$ now, and $f'(r_2) < 0$

Then $r = r_2$ is a stable limit cycle.

Also notice that $f(r) \nearrow$ for $r \gtrsim 0$, Then $r=0$ is unstable

Then, the trajectories going to. $r_0 = 0$, for $\mu < 0$, will jump to $r = r_2$ for $\mu > 0$; this is a subcritical Hopf bifurcation.

Now, if we return from $\mu > 0$ to $\mu < 0$, we still have the limit cycle $r = r_2$ stable, then the trajectories do not go to the origin, until $\mu < -\frac{1}{4}$: We have a hysteresis

In this case: $r_1^2 < 0$, $r_2^2 < 0$.
$r_1^2 \in \mathbb{C}$, $r_2^2 \in \mathbb{C}$,

and the only fixed point becomes
$r_0^2 = 0$.
which becomes stable: for $\mu < -\frac{1}{4}$


Here: $f(r_0) = 0$
and $f'(r_0) < 0$: $\Rightarrow r_0$ is stable.

This destruction is a: <u>global</u> <u>bifurcation of cycles</u>.

How to distinguish among Subcritical, Supercritical and Degenerate Bifurcation?..

Linearization does <u>not</u> provide a feeling: in any case, the eigenvalues move from $Re(\lambda) < 0$ to $Re(\lambda) > 0$

The solution given by the text is to use the computer:

(a) if the amplitude shrinks back to zero, as the parameter is reversed, then it is maybe supercritical.

(b) if it presents hysteresis, then it is maybe.

---

<u>Degenerate</u> <u>Hopf</u> <u>bifurcation</u>

<u>Example</u>: damped pendulum;

$$\ddot{x} + \mu \dot{x} + \sin x = 0.$$

for $\mu > 0$ to $\mu < 0$
stable spiral unstable spiral

At $\mu = 0$, we have a <u>band</u>, i.e., a continuum of closed orbits due to the conservation of energy.

These trajectories are <u>not</u> limit cycles, since they are <u>not</u> isolated (we require them to be isolated to be limit cycles)

<u>Degenerate</u> <u>Hopf</u> <u>bifurcation</u> arises when you have a conservative system in the background (hence, for $\mu = 0$).

02 04 05 2010.

## Example Subcritical Hopf bifurcation

$$\dot{x} = \mu x - y + x y^2$$

$$\dot{y} = x + \mu y + y^3.$$

Show there is a Hopf bifurcation occurs at $(x,y)=(0,0)$, as $\mu$ varies. Is the bifurcation, subcritical, supercritical or degenerate?

Sol'n: At the origin: the Jacobian is $J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$

$$J(x,y) = \begin{pmatrix} \mu + y^2 & -1 + 2xy \\ 1 & \mu + 3y \end{pmatrix} \xrightarrow{(0,0)}$$

Here: $\tau = 2\mu$

$\Delta = \mu^2 + 1.$

Eival's $(\mu - \lambda)^2 + 1 = 0$

$\Rightarrow \boxed{\lambda = \mu \pm i}$

Hopf bifurcation, when $\mu = \mu_c = 0$.

To decide it is supercritical, subcritical, or degenerate, Use polar coordinates.

$$x\dot{x} = \mu x^2 - xy + x^2 y^2$$

$$y\dot{y} = xy + \mu y^2 + y^4.$$

$$\boxed{\dot{r} = \mu r + r y^2}$$

$$x\dot{x} + y\dot{y} = \mu(x^2+y^2) + (x^2+y^2)y^2$$

$$\tfrac{1}{2}(x^2+y^2)^\circ = \mu(x^2+y^2) + (x^2+y^2)y^2$$

$$\tfrac{1}{2}(r^2)^\circ = \mu r^2 + r^2 y^2 \Rightarrow \tfrac{2}{2} r\dot{r} = \mu r^2 + r^2 y^2$$

* For $\mu > 0$;

$$\dot{r} = \mu r + r y^2 \geq \mu r$$

then:

$$r(t) \geq r_0 e^{\mu t}.$$

i.e. all trajectories go to infinity. } No stable limit cycle then: cannot be <u>super critical</u>

i.e. no closed curves for $\mu > 0$.

* for $\mu = 0$:

$\dot{r} = r y^2$ is still non-negative.

Then $r \nearrow$, and there is no closed trajectories,

i.e. it is <u>not</u> conservative.

Then, it is not degenerate

$\boxed{\mu > 0}$ * Then it must be subcritical, as an <u>unstable</u> limit cycle is ~~ge~~ created from going from $\mu > 0$ to $\mu < 0$.

→ | An <u>unstable</u> <u>limit</u> <u>cycle</u> surround a stable <u>fixed</u> point; this is typical for a <u>subcritical</u> Hopf bifurcation

→ { The cycle is nearly elliptical. and the cycle surrounds and surrounds a gently winding spiral } Typical in sub- or super-critical Hopf bifurcations;