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CENTER MANIFOLD THEORY. (Introduction to Applied Nonlinear Dynamical Systems and Chaos) (S. Wiggings. Text in Applied Math Vol 2.)

Simplify a dinamical system $\left\{ \begin{array}{l} \text{Method of normal forms} \\ \\ \text{Center Manifold theory} \end{array} \right.$

Motivation: for center manifold:

$$\dot{x} = Ax. \quad \text{linear system} \quad x \in \mathbb{R}^n.$$
$$A \in \mathbb{R}^n \times \mathbb{R}^n.$$

Each linear system has three subspaces, spanned by eigvcts with eival's such that:

E^s – negative real part
E^u – positive real part
E^c – zero real part

E^s – orbits starting here, decay to zero as $t \longrightarrow \infty$

E^u – orbits starting here, go to ∞ as $t \longrightarrow \infty$

E^c – orbists starting here, neither grow or decay.

If $E^u = \phi$, then any orbit will rapidly decay to E^c.

If we are interested in long-time behaviour (i.e. stability) we need to investigate the system restricted to E^c.

→ It would be nice to have a reduction principle applied to the stability of nonhyperbolic points of nonlinear systems (vector fields).

→ Namely, if there were an invariant "center manifold" passing through the fixed point to which the system can be restricted in order to study its asymptotic behaviour in the neighbourhood of the fixed point.

→ This is the spirit of the Center Manifold Theory.

$\boxed{18.1}$ Center Manifold for Vector Fields.

$$(18.1.1) \qquad \begin{aligned} \dot{x} &= Ax + f(x,y) \qquad (x,y) \in \mathbb{R}^c \times \mathbb{R}^s, \\ \dot{y} &= By + g(x,y), \end{aligned}$$

where:

$$(18.1.2) \qquad \begin{aligned} f(0,0) &= 0, \qquad Df(0,0) = 0, \\ g(0,0) &= 0, \qquad Dg(0,0) = 0, \end{aligned}$$

i.e., the origin is a fixed pt and the linear analysis fails

$A \in \mathbb{R}^{c \times c}$ matrix with $Re(\lambda) = 0$

$B \in \mathbb{R}^{s \times s}$ matrix with $Re(\lambda) < 0$

If $Re(\lambda) \neq 0$ for both eigenvalues, the fixed point is called hyperbolic.

Also $f, g \in C^r(\mathbb{R})$; $r \geq 2$.

Center Manifold Definition:

Def An invariant manifold will be called a center manifold for (18.1.1) if it can locally be represented as follows:

$$W^c(0) = \left\{ (x,y) \in \mathbb{R}^c \times \mathbb{R}^s \;\middle|\; \begin{array}{l} y \in h(x), \; |x| < \delta \\ h(0) = 0, \; Dh(0) = 0 \end{array} \right\}.$$

for δ sufficiently small.

Rmk: $h(0)=0$, $Dh(0)=0 \Rightarrow$

$\Rightarrow W^c(0)$ is tangent to E^c at $(x,y)=(0,0)$

Center Manifold: Existence.

Thm. There exists a C^r center manifold for (18.1.1). The dynamics of the system (18.1.1) restricted to the center manifold is, for u sufficiently small, given by the following c-dimensional system. (vector field):

$$\dot{u} = Au + f(u, h(u)); \quad u \in \mathbb{R}^c. \qquad (18.1.3)$$

Notation "u" The use of "u" (instead of x) is to emphasize that the restriction of the vector field to the center manifold is a vector field on a nonlinear surface.

The next results states that the dynamics of (18.1.3) near $u=0$ determines the dynamics of (18.1.1) near $(x,y)=(0,0)$.

Center Manifold : Stability: (In (i), you can change stable by asympt. stable or unstable)

(i) Suppose the zero solution of (18.1.3) is stable. Then, the zero solution of (18.1.1) is also stable.

(ii) Suppose the zero solution of (18.1.3) is stable. Then, if $(x(t), y(t))$ is a solution of (18.1.1) with $(x(0), y(0))$ sufficiently small, there is a solution $u(t)$ of (18.1.3) such that as $t \longrightarrow \infty$

$$x(t) = u(t) + O(e^{-\gamma t})$$
$$y(t) = h(u(t)) + O(e^{-\gamma t}),$$

where $\gamma > 0$ is a constant.

In words:
For initial conditions of the full system sufficiently close to the origin, trajectories through them asymptotically approach a trajectory in the center manifold.

In particular:
Equilibrium points sufficiently close to the origin, suff. small amplitude periodic orbits, as well as "small" homoclinic and heteroclinic orbits are contained in the center manifold.

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How to compute the Center Manifold so we can trap the benefits of the Center Manifold thus..

An equation for $h(x)$ should be derived, so that its graph is the center manifold for (18.1.1).

Suppose we have a center manifold:

$$W^c(0) = \{ (x,y) \in \mathbb{R}^c \times \mathbb{R}^s \,/\, y = h(x),\ |x| < \delta,\ h(0) = 0,\ Dh(0) = 0 \}.$$

δ is sufficiently small.

$W^c(0)$ is invariant under the dynamics of (18.1.1), and using this property, a quasilinear pde. for $h(x)$ is ~~derivated~~ derived.

1. If $(x,y) \in W^c(0)$, then; they should satisfy:

$$y = h(x) \quad \text{-------} \quad (18.1.5)$$

2. Diff. (18.1.5) w.r.t. time:

$$\dot{y} = Dh(x)\dot{x}. \qquad (6)$$

4. Any point on $W^c(0)$ obeys the dynamics of (18.1.1).

Then:
$$\dot{x} = Ax + f(x, h(x))$$
$$\dot{y} = Bh(x) + g(x, h(x)).$$

and substitute into (6):

$$Dh(x)\left[Ax + f(x,h(x))\right] = Bh(x) + g(x,h(x)). \qquad (9)$$

i.e.

$$N(h(x)) \equiv Dh(x)\left[Ax + f(x,h(x))\right] - Bh(x) - g(x,h(x)) = 0 \quad (10).$$

This is the (quasilinear) partial differential equation of the Center Manifold. We need to solve (9) (or (10)) to find the Center Manifold.

The original problem of solving (9) is as hard to solve (9). The following Theorem tell us how to compute an approximated solution to (10), to any degree of accuracy.

<u>Center Manifold</u>: <u>Approximation</u>.

<u>Theorem</u>: Let $\phi : \mathbb{R}^c \longrightarrow \mathbb{R}^s$ be a C^1 mapping with

$$\phi(0) = D\phi(0) = 0.$$

such that:

$$N(\phi(x)) = O(|x|^q) \quad \text{as} \quad x \longrightarrow 0$$

for some $q > 1$. Then:

$$|h(x) - \phi(x)| = O(|x|^q), \quad \text{as } x \longrightarrow 0.$$

With this thm, we can compute the Center Manifold to any degree of accuracy, by solving (10) to the same degree of accuracy. We use power series expansions.

Example: 18.1.1. Consider the vector field.

$$\dot{x} = x^2 y - x^5 \qquad (11)$$
$$\dot{y} = -y + x^2$$

The origin (0,0) is a fixed point. Is it stable or unstable?

Linearizing (11) about the origin: $\dot{\xi} = 0$, $\dot{\eta} = -\eta$

with eigenvalues $\lambda = 0$, $\lambda = -1$.

Then, linearization fails to give any conclusions for stability of (0,0). Use the Center Manifold Theory.

By theorem of existence, there is a Center Manifold (12)

$$W^c(0) = \{(x,y) \in \mathbb{R}^2 / \ y = h(x), \ |x| < \delta, \ h(0) = Dh(0) = 0\}$$

for δ small enough.

We now compute (approximate) $W^c(0)$. Assume that $h(x)$ has the form:

$$h(x) = ax^2 + bx^3 + O(x^4) \qquad (13)$$

Substitute (13) into (9) (or (10)), and equate powers of x.

The pde. for the Center Manifold is (14).

$$Dh(x)\left[Ax + f(x, h(x))\right] - Bh(x) - g(x, h(x)) = 0 \qquad (14)$$

In this example:

$$\left.\begin{aligned} A &= 0 \\ B &= -1, \\ f(x,y) &= x^2 y - x^5 \\ g(x,y) &= x^2. \end{aligned}\right\} \quad \ldots\ldots (15)$$

Substitute (13) into (14), using (15), we get.

$$Dh(x)\left[0 + x^2 y - x^5\right] + h - x^2 = 0.$$

$$h'\left(x^2 h - x^5\right) + h - x^2 = 0.$$

$$\left(2ax + 3bx^2 + \cdots\right)\left(ax^4 + bx^5 - x^5 + \cdots\right)$$
$$+ ax^2 + bx^3 - x^2 + \cdots = 0.$$

$O(x^2)$: $(a-1)x^2 = 0 \Rightarrow a = 1.$

$O(x^3)$: $bx^3 = 0 \Rightarrow b = 0.$

$O(x^4)$ holds identically

$O(x^5)$ ....

Then $h(x) = x^2 + O(x^4)$

Using Thm of existence, the vector field restricted to the center manifold is:

$$\dot{x} = x^4 + O(x^5),$$

For x small enough, $\dot{x} > 0$ for $x > 0$, or $x < 0$.

Then, $x = 0$ is unstable on the Center Manifold.

Hence, by Thm of Stability:

$(x,y) = (0,0)$ is unstable for the full NL system (11).

Remark:

The tangent space approximation does not work, in this instance.

Wrong Argument. Consider the system (11). For (x,y) close enough to $(0,0)$, the 2nd equ. implies

$$\dot{y} \cong -y$$

Then, we may say that $y \xrightarrow[t \to +\infty]{} 0$ exponentially

Then, 1st eq would become:

$$\dot{x} = -x^5,$$

which means $\dot{x} > 0$ for $x < 0$, and $\dot{x} < 0$ for $x > 0$

This would imply ~~to~~ $x = 0$ is stable, which is a wrong conclusion.

Question.

Suppose the equilibrium point is stable. Will the tangent space approximation to the center manifold also show stability?

The answer is <u>NO</u>.

Example:

$$\dot{x} = -xy - x^6$$
$$\dot{y} = -y + x^2$$

The origin $(0,0)$ is an equilibrium.
The eigenvalues of the linearized system about $(0,0)$ are

$$\lambda = 0, -1.$$

The tangent space is the x-axis.
On the tangent approximation, $y=0$:

$$\dot{x} = -x^6.$$

then, the fixed pt. would be "unstable".
However, computing the Center Manifold.

$$h(x) = x^2 + O(4).$$

Then:

$$\dot{x} = -x^3 + O(5)$$

Then, the origin is ~~un~~ <u>stable</u>.

$\boxed{18.2}$ Center Manifold with Parameters

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$$\dot{x} = Ax + f(x,y,\varepsilon).$$
$$\dot{y} = By + g(x,y,\varepsilon).$$
$$(x,y,\varepsilon) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p.$$

where:

$$f(0,0,0) = 0, \qquad Df(0,0,0) = 0,$$
$$g(0,0,0) = 0, \qquad Dg(0,0,0) = 0.$$

and some assumptions on A and B, as in (1),

f, g are C^r $(r \geq 2)$ for $(x,y,\varepsilon) = (0,0,0)$.

A, B are not allowed to depend on ε. (Why? See ahead).

This system is solved as follows.

$$\dot{x} = Ax + f(x,y,\varepsilon)$$
$$\dot{\varepsilon} = 0$$
$$\dot{y} = By + g(x,y,\varepsilon)$$

where ε is now thought as a new dependent variable.

Notice that the variables that contribute with $Re(\lambda)$ are the variables (x,ε). Then, the center manifold is now represented as

$$y = h(x,\varepsilon),$$

Then, the vector field is reduced to:

$$\dot{u} = Au + f(u, h(u,\varepsilon), \varepsilon)$$
$$\dot{\varepsilon} = 0$$

All the theory works, except that <u>bifurcations</u> might be created

Thus, since center manifold exists for $(x,\varepsilon) \sim (0,0)$, all bifurcating solutions will be contained on the lower dimensional center manifold.

Computing the center manifold works the same way:

$$W^c_{loc}(0) = \left\{ (x,\varepsilon,y) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^s \;/\; y = h(x,\varepsilon),\ |x| < \delta,\ |\varepsilon| < \bar{\delta},\ h(0,0) = 0,\ Dh(0,0) = 0 \right\}.$$

The only difference is that:

$$\dot{y} = D_x h\, \dot{x} + D_\epsilon h\, \dot{\varepsilon}, \quad \text{but } \dot{\varepsilon} = 0$$
$$= D_x h\, \dot{x}$$

and everything is reduced to the same pde. for h:

$$D_x h(x,\varepsilon)\left[Ax + f(x, h(x,\varepsilon),\varepsilon)\right] - Bh(x,\varepsilon) - g(x, h(x,\varepsilon),\varepsilon) = 0$$

<u>Remark</u> Since ε is now considered as a new <u>dependent</u> variable terms such as $x_p \varepsilon_j$ are now <u>non</u>-linear terms.
So if A depends on ε, that dependance is move to $f(x,y,\varepsilon)$
Same, if B depends on ε, nonlinear $y_i \varepsilon_j$ are moved to $g(x,y,\varepsilon)$.