Asymptotic Analysis of Integrals (Quick review)

(Applied Asymptotic Analysis)
(by Peter Miller)

Consider the integral:

$$F(\lambda) = \int_0^T e^{-\lambda t} \phi(t)\, dt, \quad \lambda > 0.$$

For general $\phi(t)$ this is almost impossible to solve in terms of elementary solutions. Some easy examples where this integral can be solved is for $\phi(t)$ polynomials or periodic functions like $\sin t$ or $\cos t$, but not in general for $t \sin t$. However, can we get some information about this integral?

In several cases, the answer to this question is "yes".

If $\phi(t)$ has the form:

$$\phi(t) = t^m g(t), \quad m \text{ - non-negative integer.}$$

then, if we let $\lambda \longrightarrow \infty$: (under some hypothesis)

(0) $$F(\lambda) = \int_0^T e^{-\lambda t} \phi(t)\, dt < \infty, \quad \forall \lambda > 0$$

(1) and $$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \frac{(m+n)!}{\lambda^{m+n+1}}, \quad \text{as } \lambda \to \infty$$

Evenmore, if $m \equiv \sigma > -1$, then. (not necessarily an integer.)

(2) $$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\, \Gamma(\sigma+n+1)}{n!\, \lambda^{\sigma+n+1}}, \quad \text{as } \lambda \to \infty$$

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where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt$ is the <u>gamma</u> function.

<u>Watson's Lemma</u> If: $T > 0$, $\sigma > -1$,

$\phi(t) = t^\sigma g(t)$ is complex-valued,

$$\int_0^T |t|^\sigma |\phi(t)|\, dt < \infty .$$

$g^{(k)}(0)$ exists for $k = 0, 1, 2, \ldots$

Then, equations (0), (1) and (2) hold (previous page).

<u>Sketch of the Proof</u>.

$$F(\lambda) = \int_0^s e^{-\lambda t} \phi(t) + \int_s^T e^{-\lambda t} \phi(t)\, dt$$

where $s < T$ is small enough:

$$\left| \int_s^T e^{-\lambda t} \phi(t)\, dt \right| \le e^{-\lambda s} \int_0^T |\phi(t)|\, dt,$$

is exponent small in λ

the integral is λ independent.

Similarly, we can show that:
se puede probar que:

$$\int_0^s e^{-\lambda t} t^\sigma g(t)\, dt = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} \int_0^s e^{-\lambda t} t^{\sigma+n}\, dt$$

$$+ \int_0^s e^{-\lambda t} t^\sigma r_N(t)\, dt$$

Now:

$$\int_0^S e^{-\lambda t} t^{\sigma} r_N(t)\, dt \leq$$

$$\leq \sup_{0\leq \tau \leq S} \left| g^{(N+1)}(\tau) \right| \cdot \frac{1}{(N+1)!} \int_0^S e^{-\lambda t} t^{\sigma+N+1}\, dt.$$

We can prove that:

$$\int_0^S e^{-\lambda t} t^{\sigma+p}\, dt = \frac{\Gamma(\sigma+p+1)}{\lambda^{\sigma+p+1}} + o\left(\lambda^{-(\sigma+p+1)}\right)$$

From where it follows that:

$$F(\lambda) \simeq \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \cdot \frac{\Gamma(\sigma+n+1)}{\lambda^{\sigma+n+1}}, \quad \text{as} \quad \lambda \longrightarrow +\infty.$$

Laplace method for Asymptotic expansions of Integrals

Integrals of the form:

$$F(\lambda) \equiv \int_0^T e^{-\lambda t} \phi(t)\, dt$$

$$F(\lambda) \equiv \int_{-\alpha}^{\beta} e^{-\lambda t^2} \phi(t)\, dt.$$

can be approximated for $\lambda > 0$, and $\lambda \longrightarrow \infty$.

The generic integral:

$$F(\lambda) = \int_a^b e^{\lambda R(t)} g(t)\, dt.$$

can be approximated.

$$F(\lambda) = e^{\lambda R_{max}} \int_a^a e^{\lambda \tilde{R}(t)} g(t)\,dt$$

So we have either:

$$F(\lambda) = e^{\lambda R_{max}} \left\{ \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \frac{d^n}{dt^n}\Big|_{t=a} \left(g(t)\,\tau'(t)\right) + \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \frac{d^n}{dt^n}\Big|_{t=b} \left(g(t)\,\tau'(t)\right) \right\}$$

$\lambda \longrightarrow \infty$

where $\tau'(t=a)$ or $\tau'(t=b)$ depends on $R(t)$,

Or:

$$F(\lambda) = \sqrt{\frac{\pi}{\lambda}}\, e^{\lambda R_{max}} \sum_{n=0}^{\infty} \frac{1}{2^{2n} n!_0 \lambda^n} \frac{d^{2n}}{dt^{2n}}\Big|_{t=t_{mid}} \left(g(t)\,(tv'+v)\right).$$

when $\lambda \longrightarrow \infty$

here $v(t)$ can be computed in terms of R.

Examples: At leading order:

~~$$F_+(\lambda) = \int_0^1 e^{\lambda(t-t^2)}\,dt \simeq$$~~

$$F(\lambda) = -\frac{e^{\lambda R_{max}} g(a)}{R'(a)} \lambda^{-1} + \frac{g(b)}{R'(b)}\lambda^{-1} + O(\lambda^{-2}),\ \text{as } \lambda \longrightarrow \infty$$

or

$$F(\lambda) = e^{\lambda R_{max}} \sqrt{\frac{-2\pi}{\lambda R''(t_{max})}}\, g(t_{max}) + O\left(\frac{1}{\lambda^{3/2}}\right)$$

as $\lambda \longrightarrow \infty$.

$-\lambda =$

Example:

$$F_-(\lambda) = \int_0^1 e^{-\lambda(t-t^2)}\, dt.$$

$$R(t) = -t + t^2, \quad g(t) \equiv 1$$

$$\max_{[0,1]} R(t) = R(0) = R(1) = 0.$$

$$R'(t) = -1 + 2t:$$

$$R'(0) = -1$$

$$R'(1) = +1.$$

$$R''(t) = +2.$$

$$F_+(\lambda) = \frac{-1}{-1}\lambda^{-1} + \frac{1}{1}\lambda + O\left(\frac{1}{\lambda^2}\right)$$

$$= +\frac{2}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad \text{as } \lambda \longrightarrow \infty.$$

Example:

$$F_+(\lambda) = \int_0^1 e^{\lambda(t-t^2)}\, dt.$$

$$R(t) = t - t^2. \qquad R'(t) = +1 - 2t$$

$$R''(t) = -2.$$

$$\max_{[0,1]} R(t) = R\left(\tfrac{1}{2}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$F_-(\lambda) = \sqrt{\frac{-2\pi}{\lambda(-2)}} \cdot 1 \cdot e^{\lambda/4} + O(\lambda^{-3/2})$$

$$= \sqrt{\frac{\pi}{\lambda}}\, e^{\lambda/4} + O\left(\frac{1}{\lambda^{3/2}}\right) \text{ as } \lambda \longrightarrow \infty$$